Specht Triangle Approximation of Large Bending Isometries

Xiang Li$^{1,2}$, and Pingbing Ming$^{1,2,*}$

$^1$LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, AMSS, Chinese Academy of Sciences, No. 55, East Road Zhong-Guan-Cun, Beijing 100190, China
$^2$School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

Abstract. We propose a Specht triangle discretization for a geometrically nonlinear Kirchhoff plate model with large bending isometries. A combination of an adaptive time-stepping gradient flow and a Newton’s method is employed to solve the ensuing nonlinear minimization problem. $\Gamma$–convergence of the Specht triangle discretization and the unconditional energy stability of the gradient flow algorithm are proved. We present several numerical examples to demonstrate that our approach significantly enhances both the computational efficiency and accuracy.

AMS subject classifications: 65N12, 65N30, 74K20
Key words: Specht triangle, plate bending, isometry constraint, adaptive time-stepping gradient flow.

1 Introduction

The geometrically nonlinear Kirchhoff plate models have drawn great attention recently because they capture the critical feature of large bending deformations of

*Emails: lixiang615@lsec.cc.ac.cn (X. Li), mpb@lsec.cc.ac.cn (P. B. Ming)
thin plates in modern nanotechnological applications [9, 20, 22, 27, 30]. The dimensional reduced nonlinear plate model has been proposed by Kirchhoff in 1850 [21], which is based on a curvature functional subject to a pointwise isometry constraint. Friesecke and Müller [18] have derived this model from three dimensional hyperelasticity via Γ− convergence. Since then, extensive studies have been carried out from the perspective of modeling and numerics, such as the single layer problem [3, 13], the bilayer problem [7, 12, 14], the thermally actuated bilayer problem [6], just to mention a few. The above models all involve minimizing an energy functional with the isometry constraint. One of the numerical difficulties is the non-convexity of the model caused by the isometry constraint [5].

In this work we focus on the single layer model, and find a deformation $y: \Omega \to \mathbb{R}^3$ of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ by minimizing

$$E[y] = \frac{1}{2} \int_\Omega |H|^2 \, dx - \int_\Omega f(x) \cdot y(x) \, dx$$

(1.1)

subject to the isometry constraint

$$(\nabla y)^\top (\nabla y) = \text{Id}_2, \text{ a.e. in } \Omega,$$

(1.2)

and Dirichlet boundary condition

$$y(x) = g, \quad \nabla y = \Phi, \quad \Phi^\top \Phi = \text{Id}_2 \quad \text{on } \Gamma_D,$$

where $H \in L^2(\Omega; \mathbb{R}^{2\times 2})$ is the second fundamental form of the parametrized surface given by

$$H_{ij} = n \cdot \partial_i \partial_j y = (\partial_i y \times \partial_j y) \cdot \partial_i \partial_j y, \quad i, j = 1, 2,$$

and $\text{Id}_2$ is the identity matrix of order 2. $\Gamma_D$ is part of the boundary of $D$. We assume that

$$g \in H^2(\Omega; \mathbb{R}^3), \quad \Phi \in H^2(\Omega; \mathbb{R}^{3\times 2}), \quad f \in L^2(\Omega, \mathbb{R}^3).$$

(1.3)

The numerical approximation of this problem consists of two parts: discretization and minimization. A proper finite element discretization needs to take into account both high order differential operator of (1.1) and the pointwise isometry constraint. In [3, 7], the authors employed a discrete Kirchhoff triangle (DKT), which was developed in [10] for the linear bending problem. DKT element possess
the degrees of freedom of gradient at the nodes, which is convenient for imposing the constraint (1.2). The implementation of DKT element is based on the construction of a discrete gradient operator that maps the gradient to another space. Although this operator makes the proof of $\Gamma$-convergence of the discrete energy more straightforward, its numerical realization is quite complicate. Moreover, DKT is not included in the standard finite element library [13]. In addition, the quadrilateral DKT element is not suitable for the adaptive grid refinement. A discontinuous Galerkin (DG) method has been employed in [12, 13] to simulate (1.1) and (1.2). DG method has been included in many existing softwares and are more flexible in imposing the boundary conditions, while it needs a careful reconstruction of the discrete energy, and a fine-tuned penalty factor.

In the present work we turn to the nonconforming finite element method, in particular we shall exploit the Specht triangle, which is an excellent nonconforming plate bending element [31], and passes all the patch tests and performs excellently, and is one of the best thin plate triangles with 9 degrees of freedom that currently available [34, Quotation from p. 345]. The Specht triangle has been systematically studied by Shi and his collaborators in [24, 28, 29, 32]. Recently the second order Specht triangle has been designed and test in [23]. The constraint (1.2) may also be imposed on the nodes, and we prove $\Gamma$-convergence of $E_h$ to $E$ in $H^1(\Omega; \mathbb{R}^3)$, which shows the correctness of the energy discretization.

To ensure the energy (1.1) decreases while maintaining the constraint (1.2), a discrete $H^2$-gradient flow method has been designed in [3, 5, 7, 12, 13] for geometrically nonlinear Kirchhoff type models. For the single layer plate, the gradient flow exhibits two attractive properties. One is the unconditional energy stability, and the other is that the $L^1$ error of the constraint (1.2) is controlled by a pseudo-time step, which is independent of the number of the iteration. However, due to the first-order nature of the gradient flow method, it usually takes a large number of iteration to lower the error committed by the constraint to a reasonable threshold. To speed up the iteration, we shall exploit an adaptive time-stepping strategy developed in [16, 25, 33] for the phase filed models. Compared to [3], our method is around five times faster without loss of accuracy. We combine the $H^2$-gradient flow and a Newton’s method to construct a global-local algorithm that improves the accuracy by around $1 \sim 2$ order of magnitude and results into a second order
method for examples with smooth minimizers. Similar idea has been used in [2] for simulation of the harmonic maps into sphere.

The remaining part of the paper is organized as follows. In Section 2, we use the Specht triangle to discretize the variational problem. The $\Gamma$-convergence of the discrete energy to the exact energy is proved in Section 3. In Section 4, we introduce a new method that combines a discrete adaptive time-stepping $H^2$-gradient flow and the Newton’s method. We present numerical experiments and test the accuracy and efficiency of the method in the last section.

2 The Specht Triangle Discretization

We fix some notations firstly. The standard notations for the Sobolev spaces, the norms and semi-norms [1] will be used. The function space $L^2(\Omega)$ consists of functions that are square integrable over $\Omega$, which is equipped with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|_{L^2(\Omega)}$, respectively. Let $H^m(\Omega)$ be the Sobolev space of square integrable functions whose weak derivatives up to order $m$ are also square integrable with the norm defined by $\|v\|_{H^m(\Omega)}^2 = \sum_{k=0}^m |v|_{H^k(\Omega)}^2$, where the semi-norm $|v|_{H^k(\Omega)} = \sum_{|\alpha|=k} \|\partial^\alpha v\|_{L^2(\Omega)}$.

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index whose components $\alpha_i$ are nonnegative integers, $|\alpha| = \alpha_1 + \alpha_2$ and $\partial^\alpha = \partial^{\alpha_1} / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2}$. For $k \geq 0$, $H^k_0(\Omega)$ is the closure in $H^k(\Omega)$ of the space of $C^\infty(\Omega)$ functions with compact supports in $\Omega$.

For $\mathrm{mes}\Gamma_D \neq 0$, the following Friedrich’s inequality holds: for any $v \in H^1(\Omega)$ that vanishes over $\Gamma_D$, there exists $C_F$ depending only on $\Gamma_D$ and $\Omega$ such that

$$
\|v\|_{L^2(\Omega)} \leq C_F \|\nabla v\|_{L^2(\Omega)}.
$$

The values of Friedrichs’ constant $C_F$ may be found in [26] for simple domains such as balls, rectangles, etc.

Under the constraint (1.2), the following identity holds pointwise:

$$
|H|^2 = |D^2 y|^2 = |\Delta y|^2,
$$
which was proved in [12, 13]. Hence, the energy functional $E[z]$ may be rewritten into
\begin{equation}
E[z] = \frac{1}{2} \int_\Omega |D^2 z(x)|^2 - \int_\Omega f(x) \cdot z(x) \, dx.
\end{equation}

We reshape the variational problem into
\begin{equation}
y = \arg\min_{z \in A} E[z],
\end{equation}
where the admissible set $A$ is defined by
\begin{equation}
A := \{ z \in \mathbb{V} \mid (\nabla z)^\top (\nabla z) = \text{Id} \text{ a.e. in } \Omega \},
\end{equation}
with
\begin{equation}
\mathbb{V} := \{ z \in H^2(\Omega; \mathbb{R}^3) \mid z(x) = g, \nabla z = \Phi \text{ on } \Gamma_D \}.
\end{equation}

### 2.1 The Specht triangle

Let $\mathcal{T}_h$ be a triangulation of $\Omega$ with maximum diameter $h$, and we assume that all the elements are shape regular in the sense of Ciarlet-Raviart [17]. We also assume that $\mathcal{T}_h$ satisfies the inverse assumption: there holds $h / h_K \leq \nu$ for $K \in \mathcal{T}_h$ and some $\nu > 0$, where $h_K$ is the diameter of the element $K$. We denote the set of all vertices of $\mathcal{T}_h$ by $\mathcal{V}_h$, the set of all edges of $\mathcal{T}_h$ by $\mathcal{E}_h$, and the set of all internal edges of $\mathcal{T}_h$ by $\mathcal{E}^m_h$. Let $\nabla_h$ be a piecewise differential operator defined for any $K \in \mathcal{T}_h$ as $(\nabla_h z)|_K = (\nabla z)|_K$.

The Specht triangle is defined by a finite element triple $(K, P_K, \Sigma_K)$ as:
\begin{align*}
&P_K = Z_K + b_K\mathbb{P}_1(K), \\
&\Sigma_K = \{ p(a_i), \partial_x p(a_i), \partial_y p(a_i), 1 \leq i \leq 3 \},
\end{align*}
where $K$ is a triangle, and $Z_K$ is the Zienkiewicz space [11] defined by
\begin{equation}
Z_K = \mathbb{P}_2(K) + \text{Span}\{ \lambda_i^2 \lambda_j - \lambda_i \lambda_j^2 \mid 1 \leq i \neq j \leq 3 \},
\end{equation}
and the element bubble function $b_K := \lambda_1 \lambda_2 \lambda_3$ with $\{\lambda_i\}_{i=1}^3$ the barycentric coordinates of $K$.

Define the finite element space $\mathbb{X}_h$ of the Specht triangle as
\begin{equation}
\mathbb{X}_h := \{ v \in H^3(\Omega) \mid v|_K \in P_K, K \in \mathcal{T}_h, v(a), \nabla v(a) \text{ are continuous for all } a \in \mathcal{V}_h \}.
\end{equation}
The interpolate operate $\Pi$ associated with $X_h$ is defined locally as $\Pi|_K = \Pi_K$ for any $v \in H^3(K)$ and

$$\Pi_K v(a) = v(a), \quad \nabla \Pi_K v(a) = \nabla v(a) \quad \text{for all } a \in V_h.$$  \hfill (2.4)

For all $v \in H^3(\Omega)$, there holds [17]:

$$\sum_{j=0}^{2} h^j \| \nabla^j (v - \Pi v) \|_{L^2(\Omega)} \leq Ch^3 \| \nabla^3 v \|_{L^2(\Omega)}. \hfill (2.5)$$

For all $k = 1, 2, 3$ and $v \in X_h$, it follows from the inverse assumption that there exists $C_{inv}$ depending on $\nu$ and $k$ but independent of $v$ such that

$$\| \nabla_h^k v \|_{L^2(\Omega)} \leq C_{inv} h^{-1} \| \nabla_h^{k-1} v \|_{L^2(\Omega)}. \hfill (2.6)$$

### 2.2 Discrete energy

We approximate (2.2) by the Specht triangle. The function in $X_h$ is continuous over the whole domain and its gradient is continuous at all nodes, which allows us to impose the isometry constraint (1.2) at each node. Define the discrete admissible set by

$$A_h := \{ z \in V_h \mid [(\nabla z)^\top \nabla z](a) = \text{Id} \text{ for all } a \in V_h \}$$

with

$$V_h := \{ z \in [X_h]^3 \mid z(a) = g(a), \nabla z(a) = \Phi(a) \text{ for all } a \in V_h \cap \Gamma_D \}.$$  

The discrete energy minimization problem reads as

$$y_h = \arg\min_{z \in A_h} E_h[z], \hfill (2.7)$$

where

$$E_h[z] = \frac{1}{2} \int_{\Omega} |\nabla_h^3 z(x)|^2 \, dx - \int_{\Omega} f(x) \cdot z(x) \, dx,$$

and $\int_{\Omega} |\nabla_h^3 z(x)|^2 = \sum_{K \in K_h} \int_K |\nabla^2 z|^2$ is in a piecewise manner.

**Proposition 2.1.** Minimization problem (2.7) has a solution.

The proof is standard, we omit it.
3 Γ-convergence of the Discrete Energy

The discrete minimization problem (2.7) is non-convex, the standard finite element error analysis fails. Instead, we prove the Γ-convergence of $E_h$ to $E$ in $H^1(\Omega; \mathbb{R}^3)$, which shows the correctness of our energy discretization. Following a standard Γ-convergence procedure in [15], we firstly show the compactness and equi-coercivity for any $z \in A_h$, and secondly, we prove the lim-sup and lim-inf inequalities. We start with the equi-coercivity of $\{E_h\}_{h \geq 0}$.

3.1 Equi-coercivity and compactness

Lemma 3.1 (Coercivity of $E_h$). Let the data $(f, g, \Phi)$ satisfy (1.3). There exists a positive constant $C$ depending on $\Omega, \Gamma_D$ and the triangulation, but independent of $h$ such that for any $z \in A_h$,

$$
\|z\|_{H^1(\Omega)} \leq C \left( \frac{E_h^{1/2}[z]}{h} + \|g\|_{H^2(\Omega)} + \|\Phi\|_{H^2(\Omega)} + \|f\|_{L^2(\Omega)} \right),
$$

and

$$
\|\nabla_h z\|_{L^2(\Omega)} \leq C \left( \frac{E_h^{1/2}[z]}{h} + \|g\|_{H^2(\Omega)} + \|\Phi\|_{H^2(\Omega)} + \|f\|_{L^2(\Omega)} \right).
$$

Compared to the known equi-coercivity results in the literature, we clarify the upper bound on the data and the discrete energy functional, and the constants $C$ in (3.1) and (3.2) may also be elucidated from the proof.

Proof. For any $z \in A_h$, let $I_h$ be the linear Lagrange interpolation operator, note the fact that

$$
I_h(z - g) \in H^1_0(\Omega; \mathbb{R}^3), \quad I_h(\nabla z - \Phi) \in H^1_0(\Omega; \mathbb{R}^{3 \times 2}).
$$

Applying the Friedrich’s inequality (2.1) to $I_h(z - g)$ and $I_h(\nabla z - \Phi)$, we obtain

$$
\|I_h(z - g)\|_{L^2(\Omega)} \leq C_F \|\nabla I_h(z - g)\|_{L^2(\Omega)},
$$

and

$$
\|I_h(\nabla z - \Phi)\|_{L^2(\Omega)} \leq C_F \|\nabla I_h(\nabla z - \Phi)\|_{L^2(\Omega)}. \tag{3.3}
$$

For $j = 0, 1$, we use the standard interpolation estimate for $I_h$ and the inverse inequality (2.6) to obtain

$$
\|\nabla^j(z - I_h z)\|_{L^2(\Omega)} \leq C_I h^{2-j} \|\nabla^2_h z\|_{L^2(\Omega)},
$$

and

$$
\|\nabla^j(z - I_h \nabla z)\|_{L^2(\Omega)} \leq C_I h^{2-j} \|\nabla^3_h z\|_{L^2(\Omega)} \leq C_I C_{inv} h^{1-j} \|\nabla^2_h z\|_{L^2(\Omega)}. \tag{3.4}
$$
Using the triangle inequality, the Friedrichs inequality (3.3) and the interpolation estimate (3.4), we obtain

\[
\| \nabla z - \Phi \|_{L^2(\Omega)} \leq \| I_h(\nabla z - \Phi) \|_{L^2(\Omega)} + \| (I - I_h)(\nabla z - \Phi) \|_{L^2(\Omega)} \\
\leq C_F \| \nabla I_h(\nabla z - \Phi) \|_{L^2(\Omega)} + C_F \| \nabla (I - I_h)(\nabla z - \Phi) \|_{L^2(\Omega)} \\
\leq C_F \| \nabla h(\nabla z - \Phi) \|_{L^2(\Omega)} + C_F \| \nabla (I - I_h)(\nabla z - \Phi) \|_{L^2(\Omega)} \\
\leq C_F \| \nabla h \|_{L^2(\Omega)} \| \nabla z - \Phi \|_{L^2(\Omega)} + C_F \| \nabla (I - I_h) \|_{L^2(\Omega)} \| \nabla z - \Phi \|_{L^2(\Omega)}.
\]

Using (3.4) with \( j = 1 \), we obtain

\[
\| \nabla (I - I_h) \|_{L^2(\Omega)} \| \nabla z \|_{L^2(\Omega)} \leq C_I C_{inv} \| \nabla h^2 \|_{L^2(\Omega)},
\]

and using (3.4) with \( j = 1 \), we obtain

\[
\| \nabla (I - I_h) \|_{L^2(\Omega)} \| \nabla \Phi \|_{L^2(\Omega)} \leq C_I h \| \nabla \Phi \|_{L^2(\Omega)}.
\]

A combination of the above three inequalities gives

\[
\| \nabla z - \Phi \|_{L^2(\Omega)} \leq (C_F + C_I h) \left( \| \nabla h^2 \|_{L^2(\Omega)} + \| \nabla \Phi \|_{L^2(\Omega)} \right) \\
+ C_F C_I C_{inv} \| \nabla h^2 \|_{L^2(\Omega)} + C_F C_I h \| \nabla \Phi \|_{L^2(\Omega)}.
\]

Hence,

\[
\| \nabla z \|_{L^2(\Omega)} \leq A \| \nabla h^2 \|_{L^2(\Omega)} + (1 + C_F)(1 + C_I h) \| \Phi \|_{H^2(\Omega)},
\]

where \( A = C_F + C_I h + C_F C_I C_{inv} \).

Proceeding along the same line that leads to the above inequality, we obtain

\[
\| z \|_{L^2(\Omega)} \leq C_F C_I h \| \nabla h^2 \|_{L^2(\Omega)} + (C_F + C_I h) \| \nabla z \|_{L^2(\Omega)} \\
+ (1 + C_F)(1 + C_I h) \| g \|_{H^2(\Omega)}.
\]

A combination of the above two inequalities leads to

\[
\| z \|_{L^2(\Omega)} \leq \tilde{A} \| \nabla h^2 \|_{L^2(\Omega)} + B \left( \| \Phi \|_{H^2(\Omega)} + \| g \|_{H^2(\Omega)} \right)
\]

(3.6)
with 

\[ \tilde{A} = (C_F + C_I h) A + C_F C_I h \quad \text{and} \quad B = (1 + C_F + C_I h)(1 + C_F)(1 + C_I h). \]

It is clear that

\[ E_h[z] \geq \frac{1}{2} \| \nabla^2_h z \|_{L^2(\Omega)}^2 - \| f \|_{L^2(\Omega)} \| z \|_{L^2(\Omega)}. \]

Using Young’s inequality, we get

\[ \| \nabla^2_h z \|_{L^2(\Omega)}^2 \leq 2 E_h[z] + 2 \| f \|_{L^2(\Omega)} \| z \|_{L^2(\Omega)} \]

\[ \leq 2 E_h[z] + \| f \|_{L^2(\Omega)} \left( A \| \nabla^2_h z \|_{L^2(\Omega)} + B \left( \| \Phi \|_{H^2(\Omega)} + \| g \|_{H^2(\Omega)} \right) \right) \]

\[ \leq 2 E_h[z] + \frac{1}{2} \| \nabla^2_h z \|_{L^2(\Omega)}^2 + 2 \tilde{A}^2 \| f \|_{L^2(\Omega)}^2 \]

\[ + 2B \| f \|_{L^2(\Omega)} \left( \| \Phi \|_{H^2(\Omega)} + \| g \|_{H^2(\Omega)} \right). \]

This immediately implies

\[ \| \nabla^2_h z \|_{L^2(\Omega)} \leq \sqrt{2 E_h[z] + 2 \tilde{A}} \| f \|_{L^2(\Omega)} + 2 \sqrt{B} \| f \|_{L^2(\Omega)}^{1/2} \left( \| \Phi \|_{H^2(\Omega)}^{1/2} + \| g \|_{H^2(\Omega)}^{1/2} \right), \]

which leads to (3.2).

Substituting the above inequality into (3.5) and (3.6), we obtain

\[ \| z \|_{H^1(\Omega)} \leq 2(A + \tilde{A}) \sqrt{E_h[z]} + 2(A + \tilde{A}) \tilde{A} \| f \|_{L^2(\Omega)} \]

\[ + 2(A + \tilde{A}) \sqrt{B} \| f \|_{L^2(\Omega)}^{1/2} \left( \| \Phi \|_{H^2(\Omega)}^{1/2} + \| g \|_{H^2(\Omega)}^{1/2} \right) \]

\[ + 2B \| f \|_{L^2(\Omega)} + \| \Phi \|_{H^2(\Omega)} + \| g \|_{H^2(\Omega)} \]

\[ \leq 2(A + \tilde{A}) \sqrt{E_h[z]} \]

\[ + 2 \left( (A + \tilde{A})^2 + B \right) \left( \| f \|_{L^2(\Omega)} + \| \Phi \|_{H^2(\Omega)} + \| g \|_{H^2(\Omega)} \right). \]

This gives (3.1).

This immediately implies

**Proposition 3.1 (Equi-coercivity).** Let the data \( (f, g, \Phi) \) satisfy (1.3). The sequence \( \{y_h\}_{h>0} \subset A_h \) has a uniform energy bound \( E_h[y_h] \leq \Lambda \), where \( \Lambda \) is independent of \( h \). Then the sequence \( \{y_h\}_{h>0} \) is uniformly bounded in \( H^1(\Omega) \), i.e.,

\[ \| y_h \|_{H^1(\Omega)} \leq C \]

with \( C \) independent of \( h \).
The uniformly boundedness of \( \{y_h\} \) in Proposition 3.1 ensures the existence of a subsequence (not relabeled) of \( \{y_h\}_h \) such that
\[
y_h \rightarrow y \quad \text{in} \quad H^1(\Omega; \mathbb{R}^3) \quad \text{as} \quad h \rightarrow 0,
\]
for some \( y \in H^1(\Omega; \mathbb{R}^3) \). We want to show that there exists \( y \in A \subset H^2(\Omega; \mathbb{R}^3) \) such that
\[
\nabla y_h \rightharpoonup \nabla y \quad \text{in} \quad L^2(\Omega; \mathbb{R}^{3 \times 2}).
\]

**Proposition 3.2.** *(Compactness in \( H^1(\Omega) \))* Let the data \((f, g, \Phi)\) satisfy (1.3). If the sequence \( \{y_h\}_{h>0} \subset A_h \) has a uniform energy bound \( E_h[y_h] \leq \Lambda \) for all \( h>0 \), where \( \Lambda \) is independent of \( h \), then there exit \( y \in A \subset H^2(\Omega; \mathbb{R}^3) \) and a subsequence (not relabeled) of \( \{y_h\}_h \) such that
\[
y_h \rightarrow y \quad \text{in} \quad H^1(\Omega; \mathbb{R}^3) \quad \text{as} \quad h \rightarrow 0.
\]

**Proof.** It follows from Lemma 3.1 and Proposition 3.1 that \( \{y_h\}_h \) and \( \{\mathcal{I}_h(\nabla y_h)\}_h \) are uniform \( H^1 \) bounded, hence there exists a weakly converging subsequence of \( \{y_h\}_h \) such that
\[
y_h \rightarrow y \quad \text{strongly in} \quad L^2(\Omega; \mathbb{R}^3),
\]
\[
\nabla y_h \rightharpoonup \nabla y \quad \text{weakly in} \quad L^2(\Omega; \mathbb{R}^{3 \times 2}),
\]
\[
\mathcal{I}_h(\nabla y_h) \rightarrow z \quad \text{strongly in} \quad L^2(\Omega; \mathbb{R}^{3 \times 2}),
\]
\[
\nabla \mathcal{I}_h(\nabla y_h) \rightharpoonup \nabla z \quad \text{weakly in} \quad L^2(\Omega; \mathbb{R}^{3 \times 2 \times 2}),
\]
for some \( y \in H^1(\Omega; \mathbb{R}^3) \) and \( z \in H^1(\Omega; \mathbb{R}^{3 \times 2}) \) as \( h \rightarrow 0 \). We apply the standard interpolation estimate for \( \mathcal{I}_h \) and the inverse inequality to get
\[
\| \nabla y_h - z \|_{L^2(\Omega)} \leq \| \nabla y_h - \mathcal{I}_h(\nabla y_h) \|_{L^2(\Omega)} + \| \mathcal{I}_h(\nabla y_h) - z \|_{L^2(\Omega)}
\]
\[
\leq C h \| \nabla^2 y_h \|_{L^2(\Omega)} + \| \mathcal{I}_h(\nabla y_h) - z \|_{L^2(\Omega)},
\]
which implies
\[
\nabla y_h \rightarrow z \quad \text{in} \quad L^2(\Omega; \mathbb{R}^{3 \times 2}).
\]
The uniqueness of \( L^2 \) weak limit implies \( \nabla y = z \), and hence \( y \in H^2(\Omega; \mathbb{R}^3) \).

Moreover, it follows from \( \nabla y_h \rightharpoonup \nabla y \) and (3.2) that
\[
\nabla y_h \top \nabla y_h \rightharpoonup \nabla y \top \nabla y \quad \text{in} \quad L^1(\Omega; \mathbb{R}^{2 \times 2}).
\]
It remains to prove that $y$ is an isometry.

Note that $I_h(\nabla y_h^\top \nabla y_h) = \text{Id}_2$. Invoking the interpolation estimate and the inverse inequality again, we obtain

$$
\|\nabla y_h^\top \nabla y_h - \text{Id}_2\|_{L^1(\Omega)} = \|\nabla y_h^\top \nabla y_h - I_h(\nabla y_h^\top \nabla y_h)\|_{L^1(\Omega)} \\
\leq C h^2 \|\nabla^2 y_h \nabla y_h\|_{L^1(\Omega)} \\
\leq C h^2 \left( \|\nabla^2 y_h\|_{L^2(\Omega)} \|\nabla y_h\|_{L^2(\Omega)} + \|\nabla^2 y_h\|_{L^2(\Omega)}^2 \right) \\
\leq C h \\
(3.7)
$$

where we have used Lemma 3.1 in the last step.

The uniqueness of the weak limit implies that $y$ is an isometry and this completes the proof.

We are ready to prove $\Gamma$-convergence of $E_h$ to $E$ in $H^1(\Omega; \mathbb{R}^3)$. Firstly we extend the domain of discrete energy functional to $H^1(\Omega; \mathbb{R}^3)$ as follows.

$$
E_h[y_h] = \left\{ \begin{array}{ll} 
E_h[y_h] & y_h \in \mathbb{A}_h, \\
\infty & y_h \in H^1(\Omega; \mathbb{R}^3) \setminus \mathbb{A}_h. 
\end{array} \right.
$$

(3.8)

**Theorem 3.1** ($\Gamma$-convergence). Let the data $(f, g, \Phi)$ satisfy (1.3). The following two properties hold

1. **Lim-inf inequality:** For every sequence $\{y_h\}_h \subset H^1(\Omega; \mathbb{R}^3)$ converging to $y$ in $H^1(\Omega; \mathbb{R}^3)$, we have
   $$
   E[y] \leq \liminf_{h \to 0} E_h[y_h].
   $$

2. **Lim-sup inequality:** For any $y \in H^1(\Omega; \mathbb{R}^3)$, there exits a sequence $\{y_h\}_h$ with $y_h \in H^1(\Omega; \mathbb{R}^3)$ such that $y_h \to y$ and
   $$
   \limsup_{h \to 0} E_h[y_h] \leq E[y].
   $$

**Proof.** We firstly prove the lim-inf inequality. Without loss of generality, we assume that $y_h \in \mathbb{A}_h \subset H^1(\Omega; \mathbb{R}^3)$ and $E_h[y_h] \leq \Lambda$, where $\Lambda$ is a positive constant independent of $h$. Otherwise, $\liminf_{h \to 0} E_h[y_h] = +\infty$, and the lim-inf inequality holds.
Proposition 3.2 implies that $y \in \mathcal{A} \subset H^2(\Omega; \mathbb{R}^3)$. We shall show

$$\nabla^2_h y_h \rightharpoonup \nabla^2 y \text{ weakly in } L^2(\Omega; \mathbb{R}^{3 \times 2 \times 2}),$$

and

$$\| \nabla^2 y \|_{L^2(\Omega)} \leq \liminf_{h \to 0} \| \nabla^2_h y_h \|_{L^2(\Omega)},$$

and

$$E[y] = \frac{1}{2} \| \nabla^2 y \|_{L^2(\Omega)}^2 - \int_{\Omega} fy \leq \liminf_{h \to 0} \left\{ \frac{1}{2} \| \nabla^2_h y_h \|_{L^2(\Omega)}^2 - \int_{\Omega} f y_h \right\} = \liminf_{h \to 0} E_h[y_h].$$

For any $\phi \in C^\infty_c(\Omega; \mathbb{R}^{3 \times 2 \times 2})$, integration by parts, we obtain

$$\int_{\Omega} \nabla^2_h y_h : \phi = \sum_{K \in T_h} \int_K \nabla^2 y_h : \phi$$

$$= - \sum_{K \in T_h} \int_K \nabla y_h : \text{div} \phi + \sum_{K \in T_h} \int_{\partial K} \nabla y_h : (\phi n)$$

$$= - \int_{\Omega} \nabla y_h : \text{div} \phi + \sum_{e \in \mathcal{E}^{\infty}_h} \int_e [\partial_n y_h] : [(\phi [n \otimes n] - C_e),$$

for any constant vector $C_e$, where we have used

$$\int_e [\partial_n y_h] = 0 \quad \text{for any edge } e$$

in the last step [28].

Denote $e = K_1 \cap K_2, K = K_1 \cup K_2$. Using the rescaled trace inequalities, we get

$$\left| \int_e [\partial_n y_h] : (\phi [n \otimes n] - C_e) \right| \leq \| [\partial_n y_h] \|_{L^2(e)} \| \phi [n \otimes n] - C_e \|_{L^2(e)}$$

$$\leq C h_{K_1}^\frac{1}{2} \| \nabla^2_h y_h \|_{L^2(K)} (h_{K_1}^{-\frac{1}{2}} \| \phi [n \otimes n] - C_e \|_{L^2(K)}$$

$$+ h_{K_1}^\frac{1}{2} \| \nabla \phi \|_{L^2(K)})$$

$$\leq C h_{K_1} \| \nabla^2_h y_h \|_{L^2(K)} \| \nabla \phi \|_{L^2(K)}.$$
Therefore, we obtain

\[
\left| \sum_{e \in \mathcal{E}^u} \int_e \frac{\partial_n y_h \cdot (\phi [n \otimes n])}{\mathrm{C} h \| \nabla^2 y_h \|_{L^2(\Omega)} \| \nabla \phi \|_{L^2(\Omega)}} \right| \leq C h \| \nabla^2 y_h \|_{L^2(\Omega)} \| \nabla \phi \|_{L^2(\Omega)}.
\]

Taking the limit \( h \to 0 \), we get

\[
\lim_{h \to 0} \int_\Omega \nabla^2 y_h : \phi = - \lim_{h \to 0} \int_\Omega \nabla y_h : \text{div} \phi + \lim_{h \to 0} \sum_{e \in \mathcal{E}^u} \int_e \frac{\partial_n y_h \cdot (\phi [n \otimes n])}{\mathrm{C} h \| \nabla^2 y_h \|_{L^2(\Omega)} \| \nabla \phi \|_{L^2(\Omega)}}
\]

\[
= - \int_\Omega \nabla y : \text{div} \phi = \int_\Omega \nabla^2 y : \phi,
\]

i.e., \( \nabla^2 y_h \rightharpoonup \nabla^2 y \) weakly in \( L^2(\Omega; \mathbb{R}^{3 \times 2 \times 2}) \).

Next we prove the lim-sup inequality. Without loss of generality, we assume that \( y \in A \subset H^2(\Omega; \mathbb{R}^3) \). The fact that smooth isometries is dense among isometries in \( H^2(\Omega)^3 \) [19] allows us to assume that \( y \in H^3(\Omega; \mathbb{R}^3) \). Let \( y_h \) be the Specht interpolant of \( y \), for all \( a \in V_h \), we get

\[
[\nabla y_h^\top \nabla y_h](a) = [\nabla y^\top \nabla y](a) = \text{Id}_2.
\]

Hence \( y_h \in A_h \). Using the interpolation estimate (2.5), we obtain \( y_h \to y \) in \( H^2(\Omega; \mathbb{R}^3) \), and whence the convergence of the discrete energy, i.e.,

\[
\lim_{h \to 0} E_h[y_h] = E[y].
\]

Combining Proposition 3.2 and Theorem 3.1, we obtain that the cluster points of the global minimizers of \( E_h \) are global minimizers of \( E \).

**Corollary 3.1.** Let data \((f, g, \Phi)\) satisfies (1.3). Let \( \{y_h\}_h \) be a sequence of almost-global minimizers of \( E_h \), i.e.

\[
E_h[y_h] \leq \inf_{w_h \in A_h} E_h[w_h] + \epsilon_h \leq C,
\]

where \( \epsilon_h \to 0 \) as \( h \to 0 \) and \( C \) is a constant independent of \( h \). Then \( \{y_h\}_h \) is uniformly bounded in the \( H^1 \) norm, and every cluster point \( y \) of \( \{y_h\} \) is a global minimizer of \( E \), i.e.

\[
E[y] = \inf_{w \in A} E[w].
\]

Moreover, there exits a subsequence of \( \{y_h\}_h \) (not relabeled) such that

\[
y_h \to y \quad \text{in} \quad H^1(\Omega) \quad \text{and} \quad \lim_{h \to 0} E_h[y_h] = E[y].
\]
4 A Combination of the Gradient Flow and Newton’s method

The $H^2$-gradient flow algorithm developed in [3] is an efficient method to solve the problem with constraint, while the first order convergence rate seems a hurdle for efficient implementation. To speed up the iteration, we firstly adopt the $H^2$-gradient flow method with an adaptive time-stepping technique, which is based on the energy variation; Such techniques appeared in the simulation of the phase field model in [25]; See also [16,33]. Secondly, we combine the gradient flow and a Newton’s method.

4.1 $H^2$-gradient flow

At a point $y_h^k \in \mathbb{X}_h^3$, we define the tangent space $F_h[y_h]$ of the admissible space $\mathbb{A}_h$ by

$$F_h[y_h] := \{ v_h \in \mathbb{X}_h^3 : v_h|_{\Gamma_D} = 0, \nabla v_h|_{\Gamma_D} = 0, \nabla v_h \nabla y_h(a) + [\nabla y_h \nabla v_h(a)] = 0 \text{ for all } a \in \mathcal{V}_h \}.$$ (4.1)

Let $\tau > 0$ be a pseudo-time parameter, $y_h^0 \in \mathbb{A}_h$ be the initial deformation, and $\epsilon$ be the stopping threshold. Define

$$d_\tau y_h^k := \frac{1}{\tau}(y_h^{k+1} - y_h^k).$$

At $(k+1)$th step, we solve

$$y_h^{k+1} \mapsto \min_{y_h^{k+1} \in y_h^k + F_h[y_h^k]} \frac{1}{2\tau} \| \nabla h^2(y_h^{k+1} - y_h^k) \|_{h}^2 + E_h[y_h^{k+1}],$$

where we have included a regularized term

$$\frac{1}{2\tau} \| \nabla h^2(y_h^{k+1} - y_h^k) \|_{h}^2$$

to ensure the energy decay. Until the stopping criterion $\| \nabla h^2 d_\tau y_h^k \|_{L^2(\Omega)} \leq \epsilon$ is met, we find $d_\tau y_h^k \in F_h[y_h^k]$ such that

$$(\nabla h^2 d_\tau y_h^k, \nabla h^2 v) + (\nabla h^2 (y_h^k + \tau d_\tau y_h^k), \nabla h^2 v) = (f, v) \quad \text{for all } v \in F_h[y_h^k].$$ (4.2)

We set

$$y_h^{k+1} := y_h^k + \tau d_\tau y_h^k.$$
Theorem 4.1. Let $y_h^{k+1}$ be the $k+1$th step solution of the gradient flow (4.2), there holds

1. Energy decay:

$$E_h[y_h^{k+1}] + \tau \sum_{n=0}^{k} \| \nabla_h^2 d_r y_n \|_h^2 \leq E_h[y_h^0].$$

(4.3)

2. Isometry violation:

$$\| \mathcal{I}_h(\nabla y_h^{k+1}) - \Id_2 \|_{L^2(\Omega)} \leq 6C^2_F(1+C_Ih)^2E_h[y_h^0] \tau.$$  

(4.4)

Proof. Choosing $v = d_r y_h^k$ in (4.2), and using the elementary identity

$$a(a-b) = \frac{1}{2}(a^2-b^2) + \frac{1}{2}(a-b)^2, \quad a, b \in \mathbb{R},$$

we get

$$E_h[y_h^{k+1}] + \frac{\tau^2 + 2\tau}{2} \| \nabla_h^2 d_r y_h^k \|_{L^2(\Omega)}^2 = E_h[y_h^k],$$

which implies

$$E_h[y_h^{k+1}] + \tau \| \nabla_h^2 d_r y_h^k \|_h^2 \leq E_h[y_h^k].$$

Using the telescoping sum, we obtain (4.3).

Note that for all $a \in V_h$, there holds

$$\| \nabla d_r y_h^k(a) \| \nabla y_h^k(a) - \Id_2 \|_{L^2(\Omega)} \leq \| \nabla d_r y_h^n(a) \|_{L^2(\Omega)} \|.$$  

On every node $a \in N_h$, we have

$$\| \nabla y_h^{k+1}(a) \| \nabla y_h^{k+1}(a) - \Id_2 \| \leq \tau^2 \sum_{n=0}^{k} \| \nabla d_r y_h^n(a) \|_{L^2(\Omega)}^2.$$
Let \(a,b,c\) be the three vertices of the element \(K\), a direct calculation gives
\[
\int_K |I_h v|^2 \, dx = \frac{|K|}{6} \left( v^2(a) + v^2(b) + v^2(c) + v(a)v(b) + v(b)v(c) + v(c)v(a) \right)
\]
\[
= \frac{|K|}{12} \left( v^2(a) + v^2(b) + v^2(c) + (v(a) + v(b) + v(c))^2 \right)
\]
\[
\geq \frac{|K|}{12} \left( v^2(a) + v^2(b) + v^2(c) \right).
\]
Hence,
\[
\|I_h (\nabla y_{k+1}^h)^\top (\nabla y_{k+1}^h) - I_d\|_{L^2(\Omega)}^2 \leq 6\tau^2 \sum_{n=0}^{k} \|\nabla I_h (\nabla d_{\tau} y_n^h)\|_{L^2(\Omega)}^2.
\]
Invoking the Friedrichs’ inequality (2.1) and the energy decay estimate (4.3), we obtain
\[
\|I_h (\nabla y_{k+1}^h)^\top (\nabla y_{k+1}^h) - I_d\|_{L^2(\Omega)}^2 \leq 6C_F^2\tau^2 \sum_{n=0}^{k} \|\nabla I_h (\nabla d_{\tau} y_n^h)\|_{L^2(\Omega)}^2
\]
\[
\leq 6C_F^2(1+C_I h)^2\tau^2 \sum_{n=0}^{k} \|\nabla^2 d_{\tau} y_n^h\|_{L^2(\Omega)}^2
\]
\[
\leq 6C_F^2(1+C_I h)^2\tau \left( E_h[y_0^h] - E_h[y_{k+1}^h] \right)
\]
\[
\leq 6C_F^2(1+C_I h)^2E_h[y_0^h] \tau.
\]

4.2 Adaptive time-stepping method

The simulation of the large bending problem needs long time computations, and the adaptive time-stepping method seems a natural choice. Theorem 4.1 implies that the \(H^2\)-gradient flow method is unconditionally stable, and the accumulation of isometry violation is related to the energy drop at each step. Therefore, we adopt a time step adaptive approach based on the energy variation to accelerate the iterations while maintaining the accuracy. Such time step adaptive strategy has been developed in [16,25,33] for the phase field models. The choice of the step size \(\tau_{k+1}\) is based on the variations of the energy
\[
\left| \frac{E_h[y_{k}^h] - E_h[y_{k-1}^h]}{\tau_k} \right|^2
\]
during the gradient flow iteration. In particular, if the energy changes smoothly, then the step size will be increased to accelerate the convergence. On the other hand, if the energy is changing rapidly, the step size will be decreased to maintain the isometry constraint.

At step \( k+1 \), we adjust the time step \( \tau_{k+1} \) as

\[
\tau_{k+1} = \max \left( \tau_{\min}, \frac{\tau_{\max}}{\sqrt{1 + \alpha \sum_{i} |E_i(y^h_k) - E_i(y^h_{k-1})|^2}} \right),
\]

where the parameters \( \tau_{\max} \) and \( \tau_{\min} \) represent the maximum and minimum step lengths, respectively, and \( \alpha \) is a case dependent parameter.

### 4.3 Newton’s method

Unlike the gradient flow method, which has global convergence properties, the Newton’s method is a local method, which achieves a second-order rate of convergence once a good initial point is chosen. Thus we apply Newton’s method with the starting point obtained by the gradient flow method. Moreover, we find numerically that Newton’s method efficiently reduces the violation error of the isometric constraints.

Three constraints of isometry \([\nabla y^h \nabla y^h](a) = \text{Id}_2\) on every node \( a \in \mathcal{V}_h \) are

\[
\begin{align*}
C^{11}(a): & \quad \partial_1 y^h(a) \cdot \partial_1 y^h(a) - 1 = 0, \\
C^{12}(a): & \quad \partial_1 y^h(a) \cdot \partial_2 y^h(a) = 0, \\
C^{22}(a): & \quad \partial_2 y^h(a) \cdot \partial_2 y^h(a) - 1 = 0.
\end{align*}
\]

We write the discrete Lagrangian as

\[
\mathcal{L}(y_h, \lambda_h) = E_h(y_h) + \sum_{a \in \mathcal{V}_h} \lambda^h_{11}(a)C^{11}(a) + \lambda^h_{12}(a)C^{12}(a) + \lambda^h_{22}(a)C^{22}(a).
\]

For every step \( k \), we add a regularized term \( \frac{1}{2\tau} \| \nabla^2_h(y^h_k - y^h_{k-1}) \|_{L^2(\Omega)}^2 \) of step length and apply the Newton’s method as

\[
\nabla^2 \left( \mathcal{L}(y^{k-1}_h, \lambda^{k-1}_h) + \frac{1}{2\tau} \| \nabla^2_h(y^k_h - y^{k-1}_h) \|_{L^2(\Omega)}^2 \right) \begin{pmatrix} \tau \cdot d_t y^k_h \\ d_t \lambda^k_h \end{pmatrix} = -\nabla \mathcal{L}(y^{k-1}_h, \lambda^{k-1}_h),
\]
and we define
\[
\begin{pmatrix}
y_h^k \\
\lambda_h^k
\end{pmatrix} = \begin{pmatrix} y_{h}^{k-1} \\
\lambda_{h}^{k-1}
\end{pmatrix} + \left( \frac{\tau \cdot d_r y_h^k}{d_r \lambda_h^k} \right) .
\]

5 Numerical Experiments

We list the basis functions of the Specht triangle \([28]\) for the sake of implementation.

For \(i = 1, 2, 3,\)
\[
\begin{align*}
\zeta_i &= \lambda_i^2 (3 - 2 \lambda_i) + 6 \sum_{i \neq j} \frac{\nabla \lambda_i \cdot \nabla \lambda_j}{|\nabla \lambda_j|^2} \phi_j, \\
\theta_i &= \lambda_i^2 (\xi_j \lambda_k - \xi_k \lambda_j) + 2(\xi_j + \xi_k) b_K (\lambda_j - \lambda_k) + 3 \frac{\nabla \lambda_i \cdot \nabla \lambda_j}{|\nabla \lambda_j|^2} \xi_j \phi_j \\
&\quad - 3 \frac{\nabla \lambda_i \cdot \nabla \lambda_k}{|\nabla \lambda_k|^2} \xi_k \phi_k, \\
\omega_i &= \lambda_i^2 (\eta_j \lambda_k - \eta_k \lambda_j) + 2(\eta_j + \eta_k) b_K (\lambda_j - \lambda_k) + 3 \frac{\nabla \lambda_i \cdot \nabla \lambda_j}{|\nabla \lambda_j|^2} \eta_j \phi_j \\
&\quad - 3 \frac{\nabla \lambda_i \cdot \nabla \lambda_k}{|\nabla \lambda_k|^2} \eta_k \phi_k, \\
\phi_i &= b_K (2 \lambda_i - 1)
\end{align*}
\]

where \(\zeta_i, \theta_i, \omega_i\) are basis functions associated with the degrees of freedom \(v(a_i), \partial_x v(a_i), \partial_y v(a_i)\), respectively.

Define
\[
I_{1h}[y_h] := \|I_h(\nabla y_h^T \nabla y_h) - \text{Id}_2\|_{L^1(\Omega)}
\]
to measure the violation of the nodal isometry constraint. It follows from Theorem 4.1 that \(I_{1h}[y_h]\) is of \(O(\tau)\). Similarly, we define
\[
I_{2h}[y_h] := \|\nabla y_h^T \nabla y_h - \text{Id}_2\|_{L^1(\Omega)},
\]
as another measurement for the violation of the isometry constraint. It follows from the estimate (3.7) that \(I_{2h}[y_h]\) is bounded by \(O(h)\). Finally, we define \(K_h[y_h]\) as the \(L^1\) error of the discrete Gaussian curvature of \(y_h:\)
\[
K_h[y_h] := \|\det(-[\nabla (I_h n_h)]^T [\nabla (I_h y_h)])\|_{L^1(\Omega)},
\]
where $n_h = \partial_1 y_h \times \partial_2 y_h$ denotes the outer normal vector of the solution surface. Under isometric condition, Gauss theorema egregium implies that Gauss curvature of every point of the exact solution should vanish.

The linear system of the $k$-th step gradient flow iteration is

$$\begin{bmatrix}(1+\tau)S & B_k^\top \\ B_k & 0\end{bmatrix}\begin{bmatrix}dY_k \\ \Lambda\end{bmatrix} = \begin{bmatrix}F - S*Y_{K-1} \\ 0\end{bmatrix},$$

(5.1)

where $S$ is the stiff matrix and $B_k$ encodes the linearized isometry constraint.

The linear system of the $k$-th step of Newton’s method is

$$\begin{bmatrix}(1+\tau)S + \tau S_\lambda & B_k^\top \\ B_k & 0\end{bmatrix}\begin{bmatrix}dY_k \\ \Lambda\end{bmatrix} = \begin{bmatrix}F - S*Y_{K-1} \\ 0\end{bmatrix} - F_\lambda,$$

(5.2)

where the two extra items $F_\lambda$ and $S_\lambda$ come from the exact isometric constraints in Lagrangian multiplier terms of $L(y_h, \lambda_h)$. We note that the cost of each iteration of the Newton’s method is similar to that of the gradient flow.

Our numerical examples are motivated by [3] and [13]. We choose $\tau_{\min} = h$, $\tau_{\max} = 10h$, $\alpha = 1e5$ and $\epsilon = 1e-3$ unless otherwise stated. We firstly consider a simple example to test the numerical performance of our method for a relatively smooth deformation.

**Example 5.1** (Rectangular plate under vertical load). Let $\Omega = (0,4) \times (0,1)$. We consider the clamped boundary condition

$$y_D(x) = (x,0)^\top \text{ and } \nabla y_D(x) = [I_2,0]^\top$$
on part of the boundary $\Gamma_D = \{0\} \times [0,1]$. We apply a constant vertical force $f(x) = (0,0,2.5 \times 10^{-2})^\top$ on the whole domain $\Omega$.

In view of Table 1, we apply the gradient flow iteration on a sequence of triangulation $T_2, T_3, T_4$ and $T_5$, and take $\tau_{\min} = h$, $\tau_{\max} = 10h$. The energy of the final configuration is $-1.58e-2$, while the convergence rate of both $I_{1h}$ and $I_{2h}$ are around 1, which is consistent with Theorem 4.1. In view of Table 5, a significant improvement in $I_{1h}$ and $I_{2h}$ has been observed with two extra Newton’s iterations. Compared to [4], the error of $I_{2h}$ is $1 \sim 2$ order of magnitude smaller, while only one sixth number of the iteration steps have been used. In view of Table 3, $I_{2h}$
Figure 1: The plate is clamped on one side and the vertical load is applied on the opposite side.

Table 1: The Specht triangle and $H^2$–gradient flow ($\tau_{\min} = h$).

<table>
<thead>
<tr>
<th>No. triangles</th>
<th>DoFs</th>
<th>$h$ step</th>
<th>$E_h$</th>
<th>$I_{1h}$</th>
<th>$I_{2h}$</th>
<th>$K_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>765</td>
<td>$2^{-2}$</td>
<td>7</td>
<td>-1.58e-2</td>
<td>3.35e-2</td>
<td>3.35e-2</td>
</tr>
<tr>
<td>512</td>
<td>2673</td>
<td>$2^{-3}$</td>
<td>11</td>
<td>-1.58e-2</td>
<td>2.12e-2</td>
<td>2.12e-2</td>
</tr>
<tr>
<td>2048</td>
<td>9945</td>
<td>$2^{-4}$</td>
<td>18</td>
<td>-1.58e-2</td>
<td>1.26e-2</td>
<td>1.26e-2</td>
</tr>
<tr>
<td>8192</td>
<td>38313</td>
<td>$2^{-5}$</td>
<td>32</td>
<td>-1.58e-2</td>
<td>7.09e-3</td>
<td>7.09e-3</td>
</tr>
</tbody>
</table>

Table 2: Newton’s method ($\tau = h$) with $y_h^0$ from Table 1 as an initial guess.

<table>
<thead>
<tr>
<th>No. triangles</th>
<th>DoFs</th>
<th>$h$ step</th>
<th>$E_h$</th>
<th>$I_{1h}$</th>
<th>$I_{2h}$</th>
<th>$K_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>765</td>
<td>$2^{-2}$</td>
<td>2</td>
<td>-1.58e-2</td>
<td>2.03e-7</td>
<td>3.32e-4</td>
</tr>
<tr>
<td>512</td>
<td>2673</td>
<td>$2^{-3}$</td>
<td>2</td>
<td>-1.58e-2</td>
<td>1.71e-7</td>
<td>8.28e-5</td>
</tr>
<tr>
<td>2048</td>
<td>9945</td>
<td>$2^{-4}$</td>
<td>2</td>
<td>-1.58e-2</td>
<td>1.65e-7</td>
<td>2.07e-5</td>
</tr>
<tr>
<td>8192</td>
<td>38313</td>
<td>$2^{-5}$</td>
<td>2</td>
<td>-1.58e-2</td>
<td>6.43e-8</td>
<td>5.19e-6</td>
</tr>
</tbody>
</table>

Table 3: Convergence rate: a combination of the gradient flow and the Newton’s method.

<table>
<thead>
<tr>
<th>$h$ $2^{-2}$</th>
<th>$2^{-3}$</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{2h}$</td>
<td>2.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
</tbody>
</table>
converges with second order, which is better than the first order rate observed in [3] and superlinear convergence observed in [13].

In what follows, we test the same example with the second order Specht triangle developed in [23] with the parameter $\alpha_1 = 18, \alpha_2 = \alpha_3 = -45$ as the spatial discretization†. It follows from Table 4 and Table 1 that the method with the second-order Specht triangle yields almost the same values for $I_{1h}$ and $I_{2h}$. Nevertheless, it is clear in Table 4 that the accuracy is improved to certain degree for $I_{1h}, I_{2h}$ and $K_h$ if we combine the Newton’s method with the $H^2$ gradient flow method.

Table 4: The 2nd order Specht triangle and $H^2$–gradient flow ($\tau_{\text{min}} = h$).

<table>
<thead>
<tr>
<th>No. triangles</th>
<th>DoFs</th>
<th>$h$ step</th>
<th>$E_h$</th>
<th>$I_{1h}$</th>
<th>$I_{2h}$</th>
<th>$K_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>1401</td>
<td>$2^{-2}$</td>
<td>7</td>
<td>-1.59e-2</td>
<td>3.35e-2</td>
<td>3.35e-2</td>
</tr>
<tr>
<td>512</td>
<td>5097</td>
<td>$2^{-3}$</td>
<td>11</td>
<td>-1.58e-2</td>
<td>2.12e-2</td>
<td>2.12e-2</td>
</tr>
<tr>
<td>2048</td>
<td>19401</td>
<td>$2^{-4}$</td>
<td>18</td>
<td>-1.58e-2</td>
<td>1.26e-2</td>
<td>1.26e-2</td>
</tr>
<tr>
<td>8192</td>
<td>75657</td>
<td>$2^{-5}$</td>
<td>32</td>
<td>-1.58e-2</td>
<td>7.09e-3</td>
<td>7.09e-3</td>
</tr>
</tbody>
</table>

Table 5: The 2nd order Specht triangle discretization with Newton’s method: $\tau = h$ and $y_h^k$ from Table 4 has been used as the initial guess.

<table>
<thead>
<tr>
<th>No. triangles</th>
<th>DoFs</th>
<th>$h$ step</th>
<th>$E_h$</th>
<th>$I_{1h}$</th>
<th>$I_{2h}$</th>
<th>$K_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>1401</td>
<td>$2^{-2}$</td>
<td>2</td>
<td>-1.58e-2</td>
<td>1.25e-7</td>
<td>3.30e-4</td>
</tr>
<tr>
<td>512</td>
<td>5097</td>
<td>$2^{-3}$</td>
<td>2</td>
<td>-1.58e-2</td>
<td>5.29e-8</td>
<td>6.41e-5</td>
</tr>
<tr>
<td>2048</td>
<td>19401</td>
<td>$2^{-4}$</td>
<td>2</td>
<td>-1.58e-2</td>
<td>1.56e-8</td>
<td>1.20e-5</td>
</tr>
<tr>
<td>8192</td>
<td>75657</td>
<td>$2^{-5}$</td>
<td>2</td>
<td>-1.58e-2</td>
<td>4.50e-9</td>
<td>2.60e-6</td>
</tr>
</tbody>
</table>

Example 5.2 (Square plate under vertical load). Let $\Omega = (0,4) \times (0,4)$. We consider the clamped boundary condition

$$y_D(x) = (x,0)^T \text{ and } \nabla y_D(x) = [I_2,0]^T$$

on part of the boundary $\Gamma_D = \{0\} \times [0,4] \cup [0,4] \times \{0\}$. We then apply a constant vertical force $f(x) = (0,0,2.5 \times 10^{-2})^T$ on the whole domain $\Omega$.

†There are many choices of the second order Specht triangle in [23]. We take the one with the best numerical performance.
Figure 2: vertical load

(a) Initial deformation $y^0_h$. (b) Deformation under the vertical load over the triangulation $T_4$.

Table 6: The Specht triangle and $H^2$–gradient flow.

<table>
<thead>
<tr>
<th>No. triangles</th>
<th>DoFs</th>
<th>$h$</th>
<th>step</th>
<th>$E_h$</th>
<th>$I_{1h}$</th>
<th>$I_{2h}$</th>
<th>$K_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>512</td>
<td>2304</td>
<td>$2^{-2}$</td>
<td>5</td>
<td>-1.01e-2</td>
<td>9.61e-3</td>
<td>1.20e-2</td>
<td>3.37e-3</td>
</tr>
<tr>
<td>2048</td>
<td>9216</td>
<td>$2^{-3}$</td>
<td>7</td>
<td>-9.69e-3</td>
<td>5.97e-3</td>
<td>8.76e-3</td>
<td>2.87e-3</td>
</tr>
<tr>
<td>8192</td>
<td>36,864</td>
<td>$2^{-4}$</td>
<td>10</td>
<td>-8.69e-3</td>
<td>3.01e-3</td>
<td>5.38e-3</td>
<td>2.05e-3</td>
</tr>
<tr>
<td>32,768</td>
<td>147,456</td>
<td>$2^{-5}$</td>
<td>16</td>
<td>-7.13e-3</td>
<td>1.27e-3</td>
<td>2.79e-3</td>
<td>1.27e-3</td>
</tr>
</tbody>
</table>

Table 7: Comparison of DKT [3], DG method [13] and our method (without Newton’s method) of the same example on the triangulation $T_5$.

<table>
<thead>
<tr>
<th>Methods</th>
<th>DoFs</th>
<th>$h$</th>
<th>step</th>
<th>$E_h$</th>
<th>$I_{2h}$</th>
<th>$K_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DKT</td>
<td>147,456</td>
<td>$2^{-5}$</td>
<td>130</td>
<td>-7.67e-3</td>
<td>3.03e-3</td>
<td>1.47e-3</td>
</tr>
<tr>
<td>DG</td>
<td>491,520</td>
<td>$2^{-5}$</td>
<td>140</td>
<td>-3.30e-3</td>
<td>2.28e-4</td>
<td>*</td>
</tr>
<tr>
<td>ours</td>
<td>147,456</td>
<td>$2^{-5}$</td>
<td>16</td>
<td>-7.13e-3</td>
<td>2.79e-3</td>
<td>1.27e-3</td>
</tr>
</tbody>
</table>

Table 8: Newton method with $y^k_h$ from Table 6 as the initial guess.

<table>
<thead>
<tr>
<th>No. triangles</th>
<th>DoFs</th>
<th>$h$</th>
<th>step</th>
<th>$E_h$</th>
<th>$I_{1h}$</th>
<th>$I_{2h}$</th>
<th>$K_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>512</td>
<td>2304</td>
<td>$2^{-2}$</td>
<td>2</td>
<td>-9.95e-3</td>
<td>5.91e-8</td>
<td>8.04e-3</td>
<td>3.30e-3</td>
</tr>
<tr>
<td>2048</td>
<td>9216</td>
<td>$2^{-3}$</td>
<td>2</td>
<td>-9.46e-3</td>
<td>3.03e-8</td>
<td>6.58e-3</td>
<td>2.76e-3</td>
</tr>
<tr>
<td>8192</td>
<td>36,864</td>
<td>$2^{-4}$</td>
<td>2</td>
<td>-8.40e-3</td>
<td>1.29e-8</td>
<td>4.39e-3</td>
<td>1.96e-3</td>
</tr>
<tr>
<td>32,768</td>
<td>147,456</td>
<td>$2^{-5}$</td>
<td>2</td>
<td>-6.89e-3</td>
<td>3.66e-9</td>
<td>2.42e-3</td>
<td>1.22e-3</td>
</tr>
</tbody>
</table>
Table 9: Convergence rate: a combination of the gradient flows and Newton’s method.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$2^{-2}$</th>
<th>$2^{-3}$</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{2h}$</td>
<td>0.29</td>
<td>0.58</td>
<td>0.86</td>
<td></td>
</tr>
</tbody>
</table>

In view of Table 6, we apply the $H^2$–gradient flow algorithm with adaptive time-stepping to a sequence of triangulations $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$ and $\mathcal{T}_5$. The convergence rates of both $I_{1h}$ and $I_{2h}$ are still around 1, and the iteration steps of our method are robust to the variation of the mesh size.

In view of Table 7, compared to [3] and [13], our adaptive strategy significantly reduces the number of the iteration. In particular on the triangulation $\mathcal{T}_5$, the DKT element in [3] required 130 iteration steps, and the DG method in [13] required 140 iteration steps, while our method achieves the same accuracy as [3] with only 16 iteration steps.

In view of Table 8 and Table 9, Newton’s method is effective in reducing $I_{1h}$. However, $I_{2h}$ remains at a similar magnitude and the convergence rate is only around 1. This could be attributed to the way the isometry constraints (1.2) is imposed, as well as the nature of the example, which is highly prone to the violation of the isometry constraints.

Another attractive example is the bulking of a plate.

**Example 5.3** (Buckling: compress a strip). Let $\Omega=(-2,2)\times(0,1)$. We impose the compressive boundary conditions

$$y_D(x)=(x_1+1.4,x_2,0)^T$$
and

$$\nabla y_D(x)=[\text{Id}_2,0]^T$$
on the two sides $\Gamma_D=\{-2,2\}\times[0,1]$, and we apply constant vertical force $f(x)=(0,0,1.0\times10^{-5})^T$ on the whole domain $\Omega$.

We choose the initial deformation $y_0^h$ as

$$y_0^h(x)=\begin{cases} 
(x_1+1.4,x_2,0), & -2 \leq x_1 \leq -1.4, \\
(0,x_2,x_1+1.4), & -1.4 \leq x_1 \leq 0, \\
(0,x_2,-x_1+1.4), & 0 \leq x_1 \leq 1.4, \\
(x_1-1.4,x_2,0), & 1.4 \leq x_1 \leq 2,
\end{cases}$$
and set
\[
\Phi = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

We plot the initial guess and the deformed state in Figure 5.

![Figure 5: Initial guess and deformed state](image)

(a) Initial guess $y^0_h$.  
(b) Buckling.

Figure 3: Buckling

Note that the initial guess is incompatible in the sense that $\|\nabla y^0_h - \Phi\|_{L^2(\Omega)} \neq 0$, which yields a relatively high initial energy $E[y^0_h]$, we hence adjust the parameters of the adaptive time-stepping to $\tau_{\text{max}} = 100\tau_{\text{min}}$ and $\alpha = 1e-4$.

By Figure 3, a bulking deformation occurs, and we display the results in Table 10, and it seems challenging to maintain $I_{1h}$ and $I_{2h}$ in this example.

### Table 10: Specht triangle, $H^2$—gradient flow: Number of triangles, degrees of freedom, number of the gradient flow iteration, energy, $L_1$-norm of the violation of isometry, $L_1$-norm of the discrete Gauss curvature.

<table>
<thead>
<tr>
<th>No. triangles</th>
<th>DoFs</th>
<th>$h$</th>
<th>step</th>
<th>$E_h$</th>
<th>$I_{1h}$</th>
<th>$I_{2h}$</th>
<th>$K_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>765</td>
<td>$2^{-2}$</td>
<td>22</td>
<td>56.35</td>
<td>35.04</td>
<td>33.88</td>
<td>1.83e-1</td>
</tr>
<tr>
<td>512</td>
<td>2304</td>
<td>$2^{-3}$</td>
<td>14</td>
<td>23.04</td>
<td>12.51</td>
<td>12.39</td>
<td>3.67e-2</td>
</tr>
<tr>
<td>2048</td>
<td>9216</td>
<td>$2^{-4}$</td>
<td>30</td>
<td>9.16</td>
<td>0.62</td>
<td>0.61</td>
<td>5.69e-3</td>
</tr>
<tr>
<td>8192</td>
<td>36,864</td>
<td>$2^{-5}$</td>
<td>75</td>
<td>9.94</td>
<td>1.35</td>
<td>1.34</td>
<td>4.72e-3</td>
</tr>
</tbody>
</table>

In Table 11, we fix the mesh size as $h = 2^{-5}$ and take $\tau_{\text{min}}$ to be $h^{1/2}$, $h$, $h^{3/2}$, respectively. It takes 367 steps to reduce $I_{2h}$ to $9.82e-2$, whereas the author in [3]
required 2,457 steps to reduce $I_{2h}$ to $6.91e-2$ with $\tau = h^{3/2}$. Next we use results $y^k_h$ from Table 10 as an initial guess of Newton’s method, and still take $\epsilon_{\text{stop}} < 1e-3$. Table 12 shows that $I_{1h}$ can be reduced to $1e-8$ and $I_{2h}$ to $1e-3$ after several additional Newton’s iteration steps.

Table 11: $H^2$- gradient flow: $h = 2^{-5}$, pseudo-time step, the number of the iteration step, $L^1$-norm of the violation of the isometry, energy.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>step</th>
<th>$I_{1h}$</th>
<th>$I_{2h}$</th>
<th>$E_h$</th>
<th>$K_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h^{1/2}$</td>
<td>18</td>
<td>1.29</td>
<td>1.29</td>
<td>10.02</td>
<td>5.25e-2</td>
</tr>
<tr>
<td>$h$</td>
<td>71</td>
<td>1.35</td>
<td>1.34</td>
<td>9.94</td>
<td>4.75e-3</td>
</tr>
<tr>
<td>$h^{3/2}$</td>
<td>367</td>
<td>9.82e-2</td>
<td>9.64e-2</td>
<td>8.66</td>
<td>2.71e-4</td>
</tr>
</tbody>
</table>

Table 12: The Specht triangle and the Newton’s method with $y^k_h$ from Table 10 as an initial guess.

<table>
<thead>
<tr>
<th>No. triangles</th>
<th>DoFs</th>
<th>$h$</th>
<th>step</th>
<th>$E_h$</th>
<th>$I_{1h}$</th>
<th>$I_{2h}$</th>
<th>$K_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>765</td>
<td>$2^{-2}$</td>
<td>46</td>
<td>8.08</td>
<td>1.61e-8</td>
<td>1.96e-1</td>
<td>7.16e-3</td>
</tr>
<tr>
<td>512</td>
<td>2304</td>
<td>$2^{-3}$</td>
<td>45</td>
<td>8.45</td>
<td>7.74e-8</td>
<td>5.25e-2</td>
<td>1.38e-3</td>
</tr>
<tr>
<td>2048</td>
<td>9216</td>
<td>$2^{-4}$</td>
<td>19</td>
<td>8.55</td>
<td>3.22e-8</td>
<td>1.34e-2</td>
<td>2.17e-4</td>
</tr>
<tr>
<td>8192</td>
<td>36,864</td>
<td>$2^{-5}$</td>
<td>37</td>
<td>8.57</td>
<td>3.56e-8</td>
<td>3.36e-3</td>
<td>1.20e-4</td>
</tr>
</tbody>
</table>

Acknowledgments

This work is supported by National Natural Science Foundation of China through Grant No. 11971467.

References


