HETEROGENEOUS MULTISCALE METHODS FOR FOURTH-ORDER SINGULAR PERTURBATIONS[*]

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Abstract. We develop a numerical homogenization method for fourth-order singular perturbation problems within the framework of heterogeneous multiscale method. These problems arise from heterogeneous strain gradient elasticity and elasticity models for architectured materials. We establish an error estimate for the homogenized solution applicable to general media and derive an explicit convergence for the locally periodic media with the fine-scale ε. For cell problems of size $\delta = \mathbb{N}\varepsilon$, the classical resonance error $\mathcal{O}(\varepsilon/\delta)$ can be eliminated due to the dominance of the higher-order operator. Despite the occurrence of boundary layer effects, discretization errors do not necessarily deteriorate for general boundary conditions. Numerical simulations corroborate these theoretical findings.

Key words. Heterogeneous multiscale method, Singular perturbation of elliptic homogenization problem, Resonance error, Boundary condition, Boundary layer effect

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1. Introduction. Consider the singular perturbations of the scalar elliptic homogenization problem [5]:

$$(1.1) \qquad \begin{cases} \iota^2 \Delta^2 u^\varepsilon - \operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega, \\ u^\varepsilon = \partial_{\boldsymbol{n}} u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $\partial_{\boldsymbol{n}}$ is the normal derivative, $0 < \varepsilon \ll 1$ is a small parameter that signifies explicitly the length scale of the heterogeneity, $0 < \iota \ll 1$ is the strength of the singular perturbations, with $\iota \to 0$ when $\varepsilon \to 0$, and the coefficient matrix A^ε belongs to a set $\mathcal{M}(\lambda, \Lambda; \Omega)$ defined by

$$(1.2) \qquad \begin{aligned} \mathcal{M}(\lambda, \Lambda; \Omega) := \Big\{ & A \in [L^\infty(\Omega)]^{d \times d} \mid A(\boldsymbol{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \lambda |\boldsymbol{\xi}|^2, \\ & A(\boldsymbol{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \frac{1}{\Lambda} |A(\boldsymbol{x})\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{x} \in \Omega \text{ and } \boldsymbol{\xi} \in \mathbb{R}^d \Big\}. \end{aligned}$$

The elements of $\mathcal{M}(\lambda, \Lambda; \Omega)$ are not necessarily symmetric. This boundary value problem represents a possible remedy of the shear bands under severe loading for the heterogeneous materials [16]. The sequence A^ε satisfying (1.2) converges to $\bar{A}$ when $\varepsilon \to 0$ in the sense of H-convergence [26], i.e., the solution u^ε of (1.1) satisfies

$$\begin{cases} u^\varepsilon \rightharpoonup \bar{u} & \text{weakly in } H_0^1(\Omega), \\ \iota \Delta u^\varepsilon \rightharpoonup 0 & \text{weakly in } L^2(\Omega), \\ A^\varepsilon \nabla u^\varepsilon \rightharpoonup \bar{A} \nabla \bar{u} & \text{weakly in } [L^2(\Omega)]^d, \end{cases}$$

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28   for some $\bar{u} \in H_0^1(\Omega)$ the weak solution of the homogenization problem

29   (1.3)
$$\begin{cases} -\mathrm{div}(\bar{A}\nabla\bar{u}) = f & \text{in } \Omega, \\ \bar{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

30   where $\bar{A} \in [L^\infty(\Omega)]^{d \times d}$ is the effective matrix. When A^ε is periodic, we refer to [38,
31   16, 36] for qualitative results on homogenization of (1.1), and the quantitative ho-
32   mogenization results can be found in [29, 31, 28].
33   Computing the full solution of (1.1) is computational intensive because one has
34   to resolves the fine-scale ε with a fourth-order conforming discretization. While a
35   sparse operator compression method for (1.1) is proposed in [18], the stiffness matrices
36   quickly become ill-conditioned even for two-dimensional problem. Instead of finding
37   a fine-scale solution, we seek an approximation of the coarse-scale solution that incurs
38   reduced computational cost without resolving the full fine-scale ε. The coarse-scale
39   solution corresponds to the H-limit (1.3).
40   To this end, we shall develop a coarse-scale simulation method within the frame-
41   work of the heterogeneous multiscale method (HMM) developed by E AND ENGQUIST
42   in [13] to solve (1.3). A comprehensive review of HMM may be found in [14, 2].
43   Recently, there have been many interesting works extending HMM to address the
44   Landau-Lifshitz equation in heterogeneous media [19, 20] and the rough-wall laminar
45   viscous flow [9], to name just a few. HMM aims to capture the macroscopic behavior
46   of a system without resolving the microscopic details. HMM consists of two key com-
47   ponents: the macroscopic solver and the cell problems for retrieving the missing data
48   of the macroscale solver. We choose the linear Lagrange finite element as the macro-
49   scopic solver because $\bar{u}$ solves a second-order elliptic boundary value problem (1.3).
50   The missing data in the macroscopic solver is the effective matrix $\bar{A}$, which is deter-
51   mined by solving certain cell problems that typically take the form of (1.1) without
52   the source term, subject to certain boundary conditions.
53   The boundary conditions of the cell problems are crucial for the accuracy and
54   efficiency of the overall method. Careful consideration of these boundary conditions
55   ensures that the macroscopic solver accurately captures the essential features of the
56   original problem. Inspired by [17], we propose four different boundary conditions for
57   the cell problem, allowing for a unified analysis. Numerical findings indicate that all
58   proposed conditions yield accurate and efficient results, with only marginal difference
59   among them. It is worth mentioning that the method is general and applicable regard-
60   less of the explicit formulation of ι and A^ε. However, the effective matrix $\bar{A}$ depends
61   on the explicit relationship between ι and ε, as demonstrated in [16, Theorem 1.3]
62   and [28].
63   We complement the method with a comprehensive analysis. Our analysis follows
64   the framework established by E, MING AND ZHANG in [15]. Since the numerical
65   effective matrix $A_H \in \mathcal{M}(\lambda, \Lambda; \Omega)$, we establish the overall accuracy in Lemma 2.5
66   which consists of the discretization error of the macroscopic solver and the error caused
67   by the approximation of the effective matrix that

68
$$\|\nabla(\bar{u} - u_H)\|_{L^2(\Omega)} \le C(\lambda, \Lambda, \bar{u})\big(H + e(\mathrm{HMM})\big),$$

69   where $e(\mathrm{HMM})$ refers to the error caused by estimating the effective matrix. Under
70   the assumption that $\iota = \mu\varepsilon^\gamma$ with $\gamma, \mu > 0$ and $A^\varepsilon(\boldsymbol{x}) = A(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$, where $A \in$
71    $[C^{0,1}(\Omega; L^\infty(\mathbb{R}^d))]^{d \times d}$ and $A(\boldsymbol{x}, \cdot)$ is $Y := [-1/2, 1/2]^d$-periodic, we analyze $e(\mathrm{HMM})$
72   in Theorem 3.14. The analysis is suitable for cell problems with general boundary

73 conditions including the essential boundary condition, the natural boundary condi-
74 tion, the free boundary condition and the periodic boundary condition. We state the
75 particular result under certain technical assumptions.

76   THEOREM 1.1 (Particular result for Theorem 3.14). *If $\iota = \mu\varepsilon^\gamma$ with $\gamma > 0$, A^ε is*
77 *periodic, and δ is an integer multiple of ε, then*

78 $$e(\text{HMM}) \leq C(\mu, A) \begin{cases} \varepsilon^{2(1-\gamma)} & 0 < \gamma < 1, \\ \varepsilon/\delta + h^2/\varepsilon^2 & \gamma = 1, \\ \varepsilon/\delta + \varepsilon^{2(\gamma-1)} + h^2/\varepsilon^2 & \gamma > 1, \|\nabla_{\boldsymbol{y}} A\|_{L^\infty(\Omega \times Y)} \text{ is bounded.} \end{cases}$$

79   Compared to the estimates of $e(\text{HMM})$ in [15, Theorem 1.2] and [12, Theorem 3.3]
80 for the second-order elliptic homogenization problem with locally periodic coefficients,
81 the theorem above includes three key terms on its right-hand side. The term $\mathcal{O}(\varepsilon/\delta)$
82 represents the resonance error, the term $\mathcal{O}(h^2/\varepsilon^2)$ denotes the discretization error
83 caused by numerically solving the cell problems [1]. The term $\mathcal{O}(\varepsilon^{2|1-\gamma|})$ captures the
84 combined effect of homogenization and singular perturbation. Notably, the theorem
85 highlights that both resonance and discretization errors vanish when $0 < \gamma < 1$ and
86 A^ε is periodic. Specifically, when $\gamma = 1$, no interaction term exists. Finally, we
87 successfully overcome the degeneracy of the discretization error for $\gamma > 1$, which
88 is caused by the emergence of the boundary layer effect [32]. These effects lead to
89 a degeneracy in the discretization error for general singular perturbation problems,
90 scaling to $\mathcal{O}(\sqrt{h/\varepsilon})$ with essential boundary conditions, as detailed in Appendix A.
91 Building upon insights from [12], we leverage the solution structure of the cell problem
92 with locally periodic coefficients to demonstrate a discretization error of $\mathcal{O}(h^2/\varepsilon^2)$.
93   Our analysis diverges from most previous work by involving different partial dif-
94 ferential equations on varying scales, necessitating novel techniques to address the
95 combined effects of singular perturbation and homogenization. To estimate $e(\text{HMM})$,
96 the key methodology in [15, 11] employs the first-order approximation of the cell
97 problem (2.3), constructed by the corrector $\boldsymbol{\chi}$, and estimates the difference within
98 this approximation; see §3. The formulation of $\boldsymbol{\chi}$ has been derived in [16]. Specifi-
99 cally, $\boldsymbol{\chi} \equiv 0$ when $0 < \gamma < 1$, suggesting that the resonance error may be eliminated
100 since the first-order approximation adheres to the boundary conditions of the cell
101 problem. Conversely, when $\gamma > 1$, the H-limit of (1.1) aligns with the second-order
102 limit, guiding us to estimate between cell problems of these two types. Our objectives
103 are twofold: firstly, to propose a unified analytical framework for various boundary
104 conditions, and secondly, to mitigate the influence of the boundary layer effect. To
105 this end, we employ the modified corrector proposed by [28]. Different treatment are
106 applied to different scenarios, ultimately producing refined results.
107   The outline of the paper is as follows. In section 2, we introduce the theory
108 of homogenization and the framework of HMM, and show the well-posedness of the
109 proposed method. In section 3, we derive the error estimate under certain assumptions
110 on ι and A^ε. In section 4, we employ nonconforming finite elements to solve the
111 cell problems and report numerical results for the problem with two-scale coefficients,
112 which are consistent with the theoretical prediction. The potential of HMM for solving
113 problems without scale separation has been demonstrated in [25]. We conclude the
114 study in section 5.
115   Throughout this paper, the constant C may differ from line to line, while it only
116 depends on the constant μ and properties for A, and it is independent of $\varepsilon, \delta, \gamma$ and
117 meshes size h.

118   **2. Heterogeneous multiscale methods.** We firstly fix some notations. The
119 space $L^2(\Omega)$ of the square-integrable functions defined on a bounded and convex
120 domain Ω is equipped with the inner product $(\cdot,\cdot)_\Omega$ and the norm $\|\cdot\|_{L^2(\Omega)}$, while
121 $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ with vanishing mean. Let $H^m(\Omega)$ be the standard
122 Sobolev space [3] with the norm $\|\cdot\|_{H^m(\Omega)}$, while $H_0^m(\Omega)$ is the closure in $H^m(\Omega)$
123 of $C_0^\infty(\Omega)$. We may drop Ω in $\|\cdot\|_{H^m(\Omega)}$ when no confusion may occur. For any
124 function f that is integrable over domain D, we denote by $\langle f \rangle_D$ the mean of f over
125 domain D.
126   In this part, we introduce a HMM-FEM to solve (1.1), and we make no assumption
127 on the explicit relation between ι and ε, and the coefficient matrix A^ε.

128   **2.1. Framework of HMM-FEM.** We employ the linear finite triangular ele-
129 ment as the macroscopic solver. Extensions to higher-order finite element macroscopic
130 solvers may be found in in [15, 12, 22]. Let X_H be the finite element space as

131
$$X_H := \left\{ Z_H \in H_0^1(\Omega) \mid Z_H|_K \in \mathbb{P}_1(K) \quad \text{for all } K \in \mathcal{T}_H \right\},$$

132 where $\mathcal{T}_H$ is a triangulation of Ω, which consists of simplices K with h_K its diameter
133 and $H := \max_{K \in \mathcal{T}_H} H_K$. We assume that $\mathcal{T}_H$ is shape-regular in the sense of Ciarlet-
134 Raviart [10]: there exists a chunkiness parameter C such that $h_K/\rho_K \leq C$, where
135 ρ_K is the diameter of the largest ball inscribed into K. We also assume that $\mathcal{T}_H$
136 satisfies the inverse assumption: there exists C such that $H/H_K \leq C$. For any
137 $Z_H \in X_H$, we define Z_l as a linear approximation of Z_H at $\boldsymbol{x}_l \in K$, i.e., $Z_l(\boldsymbol{x}) =$
138 $Z_H(\boldsymbol{x}_l) + (\boldsymbol{x} - \boldsymbol{x}_l) \cdot \nabla Z_H|_K$. The macroscopic solver aims to find $u_H \in X_H$ such that

139 (2.1)
$$a_H(u_H, Z_H) = (f, Z_H)_\Omega \quad \text{for all } Z_H \in X_H.$$

140 Here $a_H : X_H \times X_H \to \mathbb{R}$ is defined by

141 (2.2)
$$a_H(V_H, Z_H) := \sum_{K \in \mathcal{T}_H} |K| \sum_{l=1}^{L} \omega_l \nabla Z_l \cdot A_H(\boldsymbol{x}_l) \nabla V_l,$$

142 where ω_l and $\boldsymbol{x}_l$ are the quadrature weights and the quadrature nodes in K, respec-
143 tively. The quadrature scheme is assumed to be exact for linear polynomial. See;
144 e.g., [12].
145   It remains to compute $A_H(\boldsymbol{x}_l) \in \mathbb{R}^{d \times d}$. To this end, we solve

146 (2.3)
$$\iota^2 \Delta^2 v^\varepsilon - \operatorname{div}(A^\varepsilon \nabla v^\varepsilon) = 0 \qquad \text{in } I_\delta,$$

147 where the cell $I_\delta := \boldsymbol{x}_l + \delta Y$ with δ the cell size. To specify the boundary conditions
148 for (2.3), we constrain the solution as

149
$$v^\varepsilon \in H^2(I_\delta) \quad \text{and} \quad \langle \nabla v^\varepsilon \rangle_{I_\delta} = \nabla V_l.$$

150 In practice, the cell problem has to be solved numerically, and we employ conforming
151 finite element method to discretize the above cell problems. For the sake of simplicity,
152 we only consider the lowest order conforming elements such as the reduced Hsieh-
153 Clough-Tocher element [6] and the reduced Powell-Sabin element [4], among many
154 others [10, §6]. Let $\mathcal{T}_h$ be a triangulation of I_δ by the simplices with maximum mesh
155 size h, which is assumed to be shape-regular and satisfies the inverse assumption. Let
156 $X_h \subset H^2(I_\delta)$ be one such finite element space associated with the triangulation $\mathcal{T}_h$,

and we assume that there exists an interpolation operator $I_h : H^2(I_\delta) \rightarrow X_h$ such that for any $v \in H^k(I_\delta)$ with $k = 2, 3$,

$$(2.4) \qquad \| \nabla^j (I - I_h) v \|_{L^2(I_\delta)} \leq C h^{k-j} \| \nabla^k v \|_{L^2(I_\delta)} \qquad j = 0, 1, 2.$$

The existence of such interpolant may be found in [10]. It is worthwhile to mention that certain nonconforming elements such as Specht triangle [35, 21] and Nilssen-Tai-Winther element [27], are also suitable candidates to discretize the cell problem. We refer to section 4 for more discussions on the nonconforming element discretization.

We shall impose four types of boundary conditions on (2.5):

1. Essential boundary condition: $V_h = X_h \cap H_0^2(I_\delta)$;
2. Natural boundary condition: $V_h = X_h \cap H_0^1(I_\delta)$;
3. Free boundary condition: $V_h = \{ z_h \in X_h \cap L_0^2(I_\delta) \mid \langle \nabla z_h \rangle_{I_\delta} = \mathbf{0} \}$;
4. Periodic boundary condition: $V_h = X_h \cap L_0^2(I_\delta) \cap H_{\text{per}}^2(I_\delta)$.

The variational formulation for (2.3) reads as: Find $v_h^\varepsilon - V_l \in V_h$ such that

$$(2.5) \qquad a^\varepsilon(v_h^\varepsilon, z_h) = 0 \quad \text{for all } z_h \in V_h,$$

where $a^\varepsilon : H^2(I_\delta) \times H^2(I_\delta) \rightarrow \mathbb{R}$ is

$$a^\varepsilon(v, z) := (A^\varepsilon \nabla v, \nabla z)_{I_\delta} + \iota^2 (\nabla^2 v, \nabla^2 z)_{I_\delta} \quad \text{for all} \quad v, z \in H^2(I_\delta).$$

Then A_H is given by

$$(2.6) \qquad A_H(\boldsymbol{x}_l) \langle \nabla v_h^\varepsilon \rangle_{I_\delta} := \langle A^\varepsilon \nabla v_h^\varepsilon \rangle_{I_\delta}.$$

As to (2.3) with the free boundary condition, the constraint is achieved as in [37] by solving the following variational problem: Find $v_i^\varepsilon \in X_h \cap L_0^2(I_\delta)$ such that

$$(2.7) \qquad a^\varepsilon(v_i^\varepsilon, z_h) = \int_{\partial I_\delta} n_i z_h \mathrm{d}\sigma(\boldsymbol{x}) \qquad \text{for all} \quad z_h \in X_h \cap L_0^2(I_\delta),$$

and (2.6) is equivalent to

$$A_H(\boldsymbol{x}_l) := \left(\langle A^\varepsilon \nabla v_1^\varepsilon \rangle_{I_\delta} \quad \cdots \quad \langle A^\varepsilon \nabla v_d^\varepsilon \rangle_{I_\delta} \right) \left(\langle \nabla v_1^\varepsilon \rangle_{I_\delta} \quad \cdots \quad \langle \nabla v_d^\varepsilon \rangle_{I_\delta} \right)^{-1}.$$

The equivalence between the above two formulations is proved in

LEMMA 2.1. *The matrix A_H obtained by solving the cell problem (2.5) subjects to free boundary condition are equivalent with solving (2.7).*

Proof. Firstly we need to show that the matrix

$$\left(\langle \nabla v_1^\varepsilon \rangle_{I_\delta} \quad \cdots \quad \langle \nabla v_d^\varepsilon \rangle_{I_\delta} \right)$$

is nonsingular, where $v_i^\varepsilon \in X_h \cap L_0^2(I_\delta)$ is the solution of (2.7). it suffices to show that a vector $\boldsymbol{k} \in \mathbb{R}^d$ such that

$$k_1 \langle \nabla v_1^\varepsilon \rangle_{I_\delta} + k_2 \langle \nabla v_2^\varepsilon \rangle_{I_\delta} + \cdots + k_d \langle \nabla v_d^\varepsilon \rangle_{I_\delta} = \mathbf{0},$$

gives $\boldsymbol{k} = \mathbf{0}$.

We set $v_{\boldsymbol{k}}^\varepsilon = k_1 v_1^\varepsilon + k_2 v_2^\varepsilon \cdots + k_d v_d^\varepsilon$, then $\langle \nabla v_{\boldsymbol{k}}^\varepsilon \rangle_{I_\delta} = \mathbf{0}$ and $v_{\boldsymbol{k}}^\varepsilon \in V_h$. By the linear property of (2.7),

$$a^\varepsilon(v_{\boldsymbol{k}}^\varepsilon, z_h) = \sum_{i=1}^{d} k_i \int_{\partial I_\delta} n_i z_h \mathrm{d}\sigma(\boldsymbol{x}) = |I_\delta| \boldsymbol{k} \cdot \langle \nabla z_h \rangle_{I_\delta} \qquad \text{for all} \quad z_h \in X_h \cap L_0^2(I_\delta).$$

Firstly taking $z_h = v_{\boldsymbol{k}}^\varepsilon$, we obtain $v_{\boldsymbol{k}}^\varepsilon \equiv 0$. Secondly, taking $z_h = x_i - \langle x_i \rangle_{I_\delta}$, we obtain $k_i = 0$ for $i = 1, \ldots, d$, and hence $\boldsymbol{k} = \boldsymbol{0}$.

Next we let $v_{\boldsymbol{k}}^\varepsilon = k_1 v_1^\varepsilon + k_2 v_2^\varepsilon \cdots + k_d v_d^\varepsilon + \langle V_l \rangle_{I_\delta}$ such that

$$k_1 \langle \nabla v_1^\varepsilon \rangle_{I_\delta} + k_2 \langle \nabla v_2^\varepsilon \rangle_{I_\delta} + \cdots + k_d \langle \nabla v_d^\varepsilon \rangle_{I_\delta} = \nabla V_l.$$

Then $\boldsymbol{k} = (k_1, \cdots, k_d)$ exists and is unique, and $v_{\boldsymbol{k}}^\varepsilon$ solves (2.5) and

$$A_H(\boldsymbol{x}_l) \langle \nabla v_{\boldsymbol{k}}^\varepsilon \rangle_{I_\delta} = \left(\langle a^\varepsilon \nabla v_1^\varepsilon \rangle_{I_\delta} \quad \cdots \quad \langle a^\varepsilon \nabla v_d^\varepsilon \rangle_{I_\delta} \right) \left(\langle \nabla v_1^\varepsilon \rangle_{I_\delta} \quad \cdots \quad \langle \nabla v_d^\varepsilon \rangle_{I_\delta} \right)^{-1} \nabla V_l$$

$$= k_1 \langle A^\varepsilon \nabla v_1^\varepsilon \rangle_{I_\delta} + k_2 \langle A^\varepsilon \nabla v_2^\varepsilon \rangle_{I_\delta} + \cdots + k_d \langle A^\varepsilon \nabla v_d^\varepsilon \rangle_{I_\delta}$$

$$= \langle A^\varepsilon \nabla v_{\boldsymbol{k}}^\varepsilon \rangle_{I_\delta}.$$

This gives (2.6) because $\boldsymbol{k}$ is a constant vector.   □

2.2. Well-posedness of HMM-FEM. For any $z \in H^2(I_\delta)$, we define the weighted norm

$$\| z \|_\iota := \| \nabla z \|_{L^2(I_\delta)} + \iota \| \nabla^2 z \|_{L^2(I_\delta)}.$$

The wellposedness of (2.5) is included in the following lemma.

LEMMA 2.2. *The cell problem (2.5) admits a unique solution v_h^ε satisfying*

(2.8) $$\| v_h^\varepsilon \|_\iota \leq (\sqrt{\Lambda} + \sqrt{\Lambda/\lambda}) \| \nabla V_l \|_{L^2(I_\delta)}.$$

Proof. For any $z_h \in V_h$, by the Poincaré inequality, there exists a constant C_p such that

$$\| z_h \|_{H^1(I_\delta)} \leq C_p \| \nabla z_h \|_{I_\delta}.$$

Hence,

$$\| z_h \|_\iota \geq \min(1/C_p, \iota) \| z_h \|_{H^2(I_\delta)}.$$

This means that for any fixed ι, the weighted norm $\| \cdot \|_\iota$ is indeed a norm over V_h. Note that a^ε is bounded and coercive on V_h with norm $\| \cdot \|_\iota$, i.e., for any $z_h \in V_h$,

$$a_h(z_h, z_h) \geq \frac{1}{2} \min(\lambda, 1) \| z_h \|_\iota^2.$$

Hence, the cell problem (2.5) admits a unique solution by the Lax-Milgram theorem.

Next, we choose $z_h = v_h^\varepsilon - V_l \in V_h$ in (2.5) and obtain

$$a^\varepsilon(v_h^\varepsilon, v_h^\varepsilon) = (A^\varepsilon \nabla v_h^\varepsilon, \nabla V_l) \leq \| A^\varepsilon \nabla v_h^\varepsilon \|_{L^2(I_\delta)} \| \nabla V_l \|_{L^2(I_\delta)}$$

$$\leq \sqrt{\Lambda} (A^\varepsilon \nabla v_h^\varepsilon, \nabla v_h^\varepsilon)^{1/2} \| \nabla V_l \|_{L^2(I_\delta)}.$$

This immediately implies

$$(A^\varepsilon \nabla v_h^\varepsilon, \nabla v_h^\varepsilon) \leq \Lambda \| \nabla V_l \|_{L^2(I_\delta)}^2,$$

and

$$\iota^2 \| \nabla^2 v_h^\varepsilon \|_{L^2(I_\delta)}^2 \leq \sqrt{\Lambda} (A^\varepsilon \nabla v_h^\varepsilon, \nabla v_h^\varepsilon)^{1/2} \| \nabla V_l \|_{L^2(I_\delta)} \leq \Lambda \| \nabla V_l \|_{L^2(I_\delta)}^2.$$

A combination of the above two inequalities implies (2.8).   □

In the next lemma, we shall show that (2.6) satisfies the Hill's condition [17].

227   LEMMA 2.3. *There holds*

228   (2.9) $\langle \nabla z_h^\varepsilon \rangle_{I_\delta} \cdot A_H(\boldsymbol{x}_l) \langle \nabla v_h^\varepsilon \rangle_{I_\delta} = \langle \nabla z_h^\varepsilon \cdot A^\varepsilon \nabla v_h^\varepsilon \rangle_{I_\delta} + \iota^2 \langle \nabla^2 z_h^\varepsilon : \nabla^2 v_h^\varepsilon \rangle_{I_\delta},$

229   *where z_h^ε is the solution of* (2.5) *with V_l replaced by any linear function Z_l.*

230   *Proof.* It follows from $z_h^\varepsilon - Z_l \in V_h$ that

231   (2.10) $\langle \nabla z_h^\varepsilon \rangle_{I_\delta} = \nabla Z_l,$

232   and using (2.5) , we obtain

233    $a^\varepsilon(v_h^\varepsilon, z_h^\varepsilon) = a^\varepsilon(v_h^\varepsilon, Z_l) = (A^\varepsilon \nabla v_h^\varepsilon, \nabla Z_l)_{I_\delta} = |I_\delta| \nabla Z_l \cdot \langle A^\varepsilon \nabla v_h^\varepsilon \rangle_{I_\delta},$

234   which gives (2.9) with (2.6) and (2.10).   □

235   It follows from the Hill's condition that $A_H \in \mathcal{M}(\lambda, \Lambda; \Omega)$.

236   LEMMA 2.4. *There holds $A_H \in \mathcal{M}(\lambda, \Lambda; \Omega)$.*

237   *Proof.* For any $\boldsymbol{\xi} \in \mathbb{R}^d$, let $v_h^\varepsilon \in V_h$ be the solution of (2.5) with $V_l = \boldsymbol{\xi} \cdot \boldsymbol{x}$. By
238   Hill's condition (2.9),

239    $A_H(\boldsymbol{x}_l)\boldsymbol{\xi} \cdot \boldsymbol{\xi} = \langle A^\varepsilon \nabla v_h^\varepsilon \cdot \nabla v_h^\varepsilon \rangle_{I_\delta} + \iota^2 \langle |\nabla^2 v_h^\varepsilon|^2 \rangle_{I_\delta} \geq \langle A^\varepsilon \nabla v_h^\varepsilon \cdot \nabla v_h^\varepsilon \rangle_{I_\delta}.$

240   Hence,
241    $A_H(\boldsymbol{x}_l)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \lambda \langle |\nabla v_h^\varepsilon|^2 \rangle_{I_\delta} \geq \lambda |\langle \nabla v_h^\varepsilon \rangle_{I_\delta}|^2 = \lambda |\boldsymbol{\xi}|^2.$

242   On the other hand,

243    $A_H(\boldsymbol{x}_l)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \dfrac{1}{\Lambda} \langle |A^\varepsilon \nabla v_h^\varepsilon|^2 \rangle_{I_\delta} \geq \dfrac{1}{\Lambda} |\langle A^\varepsilon \nabla v_h^\varepsilon \rangle_{I_\delta}|^2 = \dfrac{1}{\Lambda} |A_H(\boldsymbol{x}_l) \langle \nabla v_h^\varepsilon \rangle_{I_\delta}|^2$

244
245    $\qquad\qquad = \dfrac{1}{\Lambda} |A_H(\boldsymbol{x}_l)\boldsymbol{\xi}|^2.$

246   A combination of the above two inequalities gives $A_H \in \mathcal{M}(\lambda, \Lambda; \Omega)$.   □

247   Lemma 2.4 gives the existence and uniqueness of u_H by the Lax-Milgram theo-
248   rem, which immediately implies the error estimate for the homogenized solution.

249   LEMMA 2.5. *There exists C independent of $\varepsilon, \iota, \delta$ and H such that*

250   (2.11) $\| \nabla(\bar{u} - u_H) \|_{L^2(\Omega)} \leq C\Big(H\| \bar{u} \|_{H^2(\Omega)} + e(\mathrm{HMM})\| f \|_{H^{-1}(\Omega)}\Big),$

251   *where*
252    $$e(\mathrm{HMM}) := \max_{K \in \mathcal{T}_H, \boldsymbol{x}_l \in K} |(\bar{A} - A_H)(\boldsymbol{x}_l)|,$$

253   *and $|\cdot|$ denotes the ℓ^2-norm of matrices and vectors.*

254   *Proof.* This is a direct consequence of [15, Theorem 1.1] and Lemma 2.4.   □

255   For general coefficients, it has been shown in [11, Lemma 2.2] that if all the
256   quadrature nodes $\boldsymbol{x}_l$ are Lebesgue points of $\bar{A}$, then

257    $$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} e(\mathrm{HMM}) = 0.$$

258   In the next section, we will derive how $e(\mathrm{HMM})$ relies on δ and ε for locally periodic
259   coefficients.

3. Error estimates for e(HMM) with locally periodic media. In this part, we estimate e(HMM) for the locally periodical coefficients A^ε. We assume that $\iota = \mu\varepsilon^\gamma$ with $\gamma > 0$ and a positive constant μ and $A^\varepsilon(\boldsymbol{x}) = A(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$ where $A \in [C^{0,1}(\Omega; L^\infty(\mathbb{R}^d))]^{d\times d}$ and $A(\boldsymbol{x}, \cdot)$ is Y-periodic. By [5, Theorem 6.3, Theorem 14.5] and [16, Theorem 1.3], the effective matrix $\bar{A}$ in (1.3) is given by

$$\bar{A}(\boldsymbol{x}) = \int_Y (A + A\nabla_{\boldsymbol{y}}^\top \boldsymbol{\chi})(\boldsymbol{x}, \boldsymbol{y})\mathrm{d}\boldsymbol{y},$$

where the corrector $\boldsymbol{\chi} = \boldsymbol{0}$ when $0 < \gamma < 1$. When $\gamma = 1$, the corrector $\boldsymbol{\chi} \in [L^\infty(\Omega; H^2(Y))]^d$ satisfies

$$\begin{cases} \mu^2\Delta_{\boldsymbol{y}}^2\chi_j - \mathrm{div}_{\boldsymbol{y}}(A\nabla_{\boldsymbol{y}}\chi_j) = \mathrm{div}_{\boldsymbol{y}}\boldsymbol{a}_j & \text{in } Y, \\ \chi_j(\boldsymbol{x}, \cdot) \text{ is } Y\text{-periodic} \quad \langle\chi_j\rangle_Y = 0, \end{cases}$$

where $\boldsymbol{a}_j$ is the j-th column of A with $A = [\boldsymbol{a}_1, \boldsymbol{a}_2 \ldots, \boldsymbol{a}_d]$.

When $\gamma > 1$, the corrector $\boldsymbol{\chi} \in [L^\infty(\Omega; H^1(Y))]^d$ satisfies

$$\begin{cases} -\mathrm{div}_{\boldsymbol{y}}(A\nabla_{\boldsymbol{y}}\chi_j) = \mathrm{div}_{\boldsymbol{y}}\boldsymbol{a}_j & \text{in } Y, \\ \chi_j(\boldsymbol{x}, \cdot) \text{ is } Y\text{-periodic} \quad \langle\chi_j\rangle_Y = 0. \end{cases}$$

3.1. Preliminary. Denote by v_l^ε the solution of (2.5) with A^ε replaced by $A_l^\varepsilon := A^\varepsilon(\boldsymbol{x}_l, \cdot/\varepsilon)$; i.e., find $v_l^\varepsilon - V_l \in V_h$ such that

(3.1)
$$a_l^\varepsilon(v_l^\varepsilon, z_h) = 0 \qquad \text{for all} \quad z_h \in V_h,$$

where a_l^ε is the same with a^ε provided that A^ε replaced by A_l^ε. The first order approximation of v_l^ε is

$$V_l^\varepsilon(\boldsymbol{x}) := V_l(\boldsymbol{x}) + \varepsilon\boldsymbol{\chi}^\gamma(\boldsymbol{x}_l, \boldsymbol{x}/\varepsilon)\cdot\nabla V_l,$$

where $\boldsymbol{\chi}^\gamma := \boldsymbol{\chi} = \boldsymbol{0}$ when $0 < \gamma < 1$, and $\boldsymbol{\chi}^\gamma \in [L^\infty(\Omega; H^2(Y))]^d$ is the solution of

(3.2)
$$\begin{cases} \mu^2\varepsilon^{2(\gamma-1)}\Delta_{\boldsymbol{y}}^2\chi_j^\gamma - \mathrm{div}_{\boldsymbol{y}}(A\nabla_{\boldsymbol{y}}\chi_j^\gamma) = \mathrm{div}_{\boldsymbol{y}}\boldsymbol{a}_j & \text{in } Y, \\ \chi_j^\gamma(\boldsymbol{x}, \cdot) \text{ is } Y\text{-periodic} \quad \langle\chi_j^\gamma\rangle_Y = 0, \end{cases}$$

when $\gamma \geq 1$. This corrector is a modification of that defined in [28, §3]. Moreover, there exists C independent of ε such that

(3.3)
$$\begin{aligned} \|\boldsymbol{\chi}^\gamma\|_{L^\infty(\Omega; H^1(Y))} + \varepsilon^{\gamma-1}\|\nabla_{\boldsymbol{y}}^2\boldsymbol{\chi}^\gamma\|_{L^\infty(\Omega; L^2(Y))} \\ + \varepsilon^{2(\gamma-1)}\|\nabla_{\boldsymbol{y}}^3\boldsymbol{\chi}^\gamma\|_{L^\infty(\Omega; L^2(Y))} \leq C. \end{aligned}$$

If A_l^ε is smoother, then (3.3) may be improved and $\boldsymbol{\chi}^\gamma \to \boldsymbol{\chi}$ when $\gamma \to \infty$ as shown below.

LEMMA 3.1. *If $\gamma > 1$ and $\|\nabla_{\boldsymbol{y}}A\|_{L^\infty(\Omega\times Y)}$ is bounded, then*

(3.4)
$$\|\nabla_{\boldsymbol{y}}(\boldsymbol{\chi}^\gamma - \boldsymbol{\chi})\|_{L^\infty(\Omega; L^2(Y))} \leq C\varepsilon^{\gamma-1},$$

and

(3.5)
$$\|\boldsymbol{\chi}^\gamma\|_{L^\infty(\Omega; H^2(Y))} + \varepsilon^{\gamma-1}\|\nabla_{\boldsymbol{y}}^3\boldsymbol{\chi}^\gamma\|_{L^\infty(\Omega; L^2(Y))} \leq C.$$

289   *Proof.* If $\gamma > 1$ and $\| \nabla_{\boldsymbol{y}} A \|_{L^\infty(\Omega \times Y)}$ is bounded, then $\| \boldsymbol{\chi} \|_{L^\infty(\Omega; H^2(Y))}$ is also
290   bounded. We rewrite (3.2) as

291   (3.6)
$$\begin{cases} \mu^2 \varepsilon^{2(\gamma-1)} \Delta_{\boldsymbol{y}}^2 \chi_j^\gamma = \operatorname{div}_{\boldsymbol{y}}(A \nabla_{\boldsymbol{y}}(\chi_j^\gamma - \chi_j)) & \text{in } Y, \\ \chi_j^\gamma(\boldsymbol{x}, \cdot) \text{ is } Y\text{-periodic} & \langle \chi_j^\gamma \rangle_Y = 0. \end{cases}$$

292   Multiplying both sides of (3.6) by $z = \chi_j^\gamma - \chi_j$, integration by parts, we obtain

293    $\| \nabla_{\boldsymbol{y}}(\chi_j^\gamma - \chi_j) \|_{L^\infty(\Omega; L^2(Y))}^2 + \varepsilon^{2(\gamma-1)} \| \nabla_{\boldsymbol{y}}^2 \chi_j^\gamma \|_{L^\infty(\Omega; L^2(Y))}^2$

294
295    $$\leq C \varepsilon^{2(\gamma-1)} \| \Delta_{\boldsymbol{y}} \chi_j^\gamma \|_{L^\infty(\Omega; L^2(Y))} \| \Delta_{\boldsymbol{y}} \chi_j \|_{L^\infty(\Omega; L^2(Y))}.$$

296   This gives (3.4) and the H^2-estimate in (3.5).
297   Finally, by the H^3-estimate of (3.6) and (1.2),

298    $$\| \chi_j \|_{L^\infty(\Omega; H^3(Y))} \leq C \varepsilon^{2(1-\gamma)} \| \operatorname{div}_{\boldsymbol{y}}(A \nabla_{\boldsymbol{y}}(\chi_j^\gamma - \chi_j)) \|_{L^\infty(\Omega; H^{-1}(Y))}$$

299
300    $$\leq C \varepsilon^{2(1-\gamma)} \| \nabla_{\boldsymbol{y}}(\chi_j^\gamma - \chi_j) \|_{L^\infty(\Omega; L^2(Y))},$$

301   which together with (3.4) gives the H^3-estimate in (3.5).   □

302   Next, we define

303   (3.7)
$$\bar{A}^\gamma(\boldsymbol{x}) := \int_Y (A + A \nabla_{\boldsymbol{y}}^\top \boldsymbol{\chi}^\gamma)(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{y}.$$

304   If $0 < \gamma \leq 1$, then $\bar{A}^\gamma = \bar{A}$.
305   If $\gamma > 1$ and $\| \nabla_{\boldsymbol{y}} A \|_{L^\infty(\Omega \times Y)}$ is bounded, invoking [28, Lemma 3.2], then we get

306   (3.8)
$$\| \bar{A}^\gamma - \bar{A} \|_{L^\infty(\Omega)} \leq C \varepsilon^{2(\gamma-1)}.$$

307   Let $\kappa := \lfloor \delta/\varepsilon \rfloor$. By (3.2) and (3.7), a direct calculation gives

308   (3.9)
$$\langle \nabla V_l^\varepsilon \rangle_{I_{\kappa\varepsilon}} = \nabla V_l \quad \text{and} \quad \langle A_l^\varepsilon \nabla V_l^\varepsilon \rangle_{I_{\kappa\varepsilon}} = \bar{A}^\gamma(\boldsymbol{x}_l) \nabla V_l.$$

309   Define a cut-off function $\rho^\varepsilon \in C_0^\infty(I_\delta)$ satisfying $0 \leq \rho^\varepsilon \leq 1$ and

310   (3.10) $\rho^\varepsilon = 1$ in $I_{\delta-2\varepsilon}$, $\rho^\varepsilon = 0$ in I_δ^ε, $|\nabla^i \rho^\varepsilon| \leq C \varepsilon^{-i}$ for $i = 1, 2, 3$,

311   where $I_\delta^\varepsilon := I_\delta \backslash I_{\delta-\varepsilon}$.
312   Using the identity (3.9), we decompose $e(\mathrm{HMM})$ into

313   (3.11)
$$(\bar{A} - A_H)(\boldsymbol{x}_l) \nabla V_l = (\bar{A} - \bar{A}^\gamma)(\boldsymbol{x}_l) \nabla V_l + \left(\langle A_l^\varepsilon \nabla V_l^\varepsilon \rangle_{I_{\kappa\varepsilon}} - \langle A_l^\varepsilon \nabla V_l^\varepsilon \rangle_{I_\delta} \right)$$
$$+ \langle A_l^\varepsilon \nabla \theta_l^\varepsilon \rangle_{I_\delta} + \left(\langle A_l^\varepsilon \nabla v_l^\varepsilon \rangle_{I_\delta} - \langle A^\varepsilon \nabla v_h^\varepsilon \rangle_{I_\delta} \right),$$

314   where the corrector $\theta_l^\varepsilon := V_l^\varepsilon - v_l^\varepsilon$.
315   We shall bound the terms in the right-hand side of (3.11) in a series of lemmas.
316   The first lemma compares the average of the flux over cells of different sizes.
       LEMMA 3.2.

317
$$|\langle A_l^\varepsilon \nabla V_l^\varepsilon \rangle_{I_{\kappa\varepsilon}} - \langle A_l^\varepsilon \nabla V_l^\varepsilon \rangle_{I_\delta}| \leq C \frac{\varepsilon}{\delta} |\nabla V_l|.$$

318   *Proof.* If $0 < \gamma < 1$, then

319                    $$\| \nabla V_l^\varepsilon \|_{L^2(I_\delta^\varepsilon)} = \| \nabla V_l \|_{L^2(I_\delta^\varepsilon)} = |I_\delta^\varepsilon|^{1/2}|\nabla V_l|.$$

320   If $\gamma \geq 1$, then we use (3.3) and obtain

321        $$\| \nabla V_l^\varepsilon \|_{L^2(I_\delta^\varepsilon)} = \| I + \nabla_{\boldsymbol{y}}\boldsymbol{\chi}^\gamma(\boldsymbol{x}_l, \cdot/\varepsilon) \|_{L^2(I_\delta^\varepsilon)}|\nabla V_l| \leq C|I_\delta^\varepsilon|^{1/2}|\nabla V_l|.$$

322   A direct calculation gives

323    $$|\langle A_l^\varepsilon \nabla V_l^\varepsilon \rangle_{I_{\kappa\varepsilon}} - \langle A_l^\varepsilon \nabla V_l^\varepsilon \rangle_{I_\delta}| \leq \left(1 - \frac{|I_{\kappa\varepsilon}|}{|I_\delta|}\right)\left(|\langle A_l^\varepsilon \nabla V_l^\varepsilon \rangle_{I_{\kappa\varepsilon}}| + |\langle A_l^\varepsilon \nabla V_l^\varepsilon \rangle_{I_\delta \setminus I_{\kappa\varepsilon}}|\right)$$

324    $$\leq \frac{|I_\delta^\varepsilon|}{|I_\delta|}|\bar{A}^\gamma(\boldsymbol{x}_l)\nabla V_l| + \Lambda\frac{\sqrt{|I_\delta^\varepsilon|}}{|I_\delta|}\| \nabla V_l^\varepsilon \|_{L^2(I_\delta^\varepsilon)}$$

325
326    $$\leq C\frac{\varepsilon}{\delta}|\nabla V_l|.$$

327   This finishes the proof.   □

328   We shall frequently used the following perturbation estimates.

329   LEMMA 3.3. *If $A \in [C^{0,1}(\Omega; L^\infty(\mathbb{R}^d))]^{d \times d}$, then*

330   (3.12) $$\| v_l^\varepsilon - v_h^\varepsilon \|_\iota \leq C\delta\| \nabla V_l \|_{L^2(I_\delta)},$$

331   *and*

332   (3.13) $$|\langle A_l^\varepsilon \nabla v_l^\varepsilon \rangle_{I_\delta} - \langle A^\varepsilon \nabla v_h^\varepsilon \rangle_{I_\delta}| \leq C\delta|\nabla V_l|.$$

333   *Proof.* Let $z = v_l^\varepsilon - v_h^\varepsilon \in V_h$ in (3.1), we obtain

334        $$a_l^\varepsilon(v_l^\varepsilon - v_h^\varepsilon, v_l^\varepsilon - v_h^\varepsilon) = ((A^\varepsilon - A_l^\varepsilon)\nabla v_h^\varepsilon, \nabla(v_l^\varepsilon - v_h^\varepsilon))_{I_\delta}.$$

335   The estimate (3.12) follows from the above identity and the fact $|A^\varepsilon - A_l^\varepsilon| \leq C\delta$.
336   Using (2.8) and (3.12), we obtain

337        $$|\langle A_l^\varepsilon \nabla v_l^\varepsilon \rangle_{I_\delta} - \langle A^\varepsilon \nabla v_h^\varepsilon \rangle_{I_\delta}| = |\langle A_l^\varepsilon \nabla(v_l^\varepsilon - v_h^\varepsilon)\rangle_{I_\delta} + \langle(A_l^\varepsilon - A^\varepsilon)\nabla v_h^\varepsilon\rangle_{I_\delta}|$$

338    $$\leq \frac{C}{\sqrt{|I_\delta|}}(\| v_l^\varepsilon - v_h^\varepsilon \|_\iota + \delta\| v_h^\varepsilon \|_\iota)$$

339
340    $$\leq C\delta|\nabla V_l|.$$

341   This gives (3.13).   □

342   To estimate the corrector, we define w_l^ε as the adjoint of v_l^ε: Find $w_l^\varepsilon - W_l \in V_h$
343   such that

344   (3.14) $$a_l^\varepsilon(z_h, w_l^\varepsilon) = 0 \quad \text{for all } z_h \in V_h,$$

345   and W_l^ε is defined the same with V_l^ε except that at the moment $\boldsymbol{\chi}^\gamma$ is the solution
346   of (3.2) with A replacing by $A^\top$.

347   **3.2. Estimate of the corrector when $0 < \gamma < 1$.** In what follows, we estimate
348 the corrector $\langle A_l^\varepsilon \nabla \theta_l^\varepsilon \rangle_{I_\delta}$, and we start with a trace inequality.

349   LEMMA 3.4. *If $z \in H^1(I_\delta)$ and $\langle z \rangle_{I_\delta} = 0$, then*

350 (3.15) $$\| z \|_{L^2(I_\delta^\xi)} \le C\sqrt{\varepsilon\delta}\| \nabla z \|_{L^2(I_\delta)}.$$

351   *Proof.* We apply the scaling $\hat{\boldsymbol{x}} := \boldsymbol{x}/\delta$ to I_δ so that the rescaled cell has diameter
352 1. Moreover, denote by $\hat{\varepsilon} := \varepsilon/\delta$ and $\hat{z}(\hat{\boldsymbol{x}}) := z(\boldsymbol{x})$, it is clear $\langle \hat{z} \rangle_{I_1} = 0$. Using the
353 trace inequality [30, Lemma 1.5] and the Poincaré inequality, we obtain

354           $$\| \hat{z} \|_{L^2(I_1^{\hat{\varepsilon}})} \le C\sqrt{\hat{\varepsilon}}\| \hat{z} \|_{L^2(I_1)}^{1/2}\| \hat{z} \|_{H^1(I_1)}^{1/2} \le C\sqrt{\hat{\varepsilon}}\| \nabla\hat{z} \|_{L^2(I_1)},$$

355 and

356      $$\| z \|_{L^2(I_\delta^\xi)} \le C\delta^{d/2}\| \hat{z} \|_{L^2(I_1^{\hat{\varepsilon}})} \le C\sqrt{\hat{\varepsilon}}\delta^{d/2}\| \nabla\hat{z} \|_{L^2(I_1)} \le C\sqrt{\hat{\varepsilon}}\delta\| \nabla z \|_{L^2(I_\delta)}.$$

357 This gives (3.15). $\square$

358   If $0 < \gamma < 1$, then $V_l^\varepsilon = V_l$ and $\theta_l^\varepsilon = V_l - v_l^\varepsilon \in V_h$.

359   LEMMA 3.5. *If $\iota = \mu\varepsilon^\gamma$ with $0 < \gamma < 1$, then there exists C such that*

360 (3.16) $$\| \theta_l^\varepsilon \|_\iota \le C\varepsilon^{1-\gamma}\| \nabla V_l \|_{L^2(I_\delta)}.$$

361   *Proof.* Choosing $z = \theta_l^\varepsilon \in V_h$ in (3.1) and using $\langle \nabla\theta_l^\varepsilon \rangle_{I_\delta} = 0$ and the fact that
362 $\bar{A}(\boldsymbol{x}_l)\nabla V_l$ is a constant vector, we obtain

363 (3.17)
$$\begin{aligned} a_l^\varepsilon(\theta_l^\varepsilon, \theta_l^\varepsilon) = a_l^\varepsilon(V_l, \theta_l^\varepsilon) &= \left((A_l^\varepsilon - \bar{A}(\boldsymbol{x}_l))\nabla V_l, \nabla\theta_l^\varepsilon\right)_{I_\delta} \\ &= \left((A_l^\varepsilon - \bar{A}(\boldsymbol{x}_l))\nabla V_l, \rho^\varepsilon\nabla\theta_l^\varepsilon\right)_{I_\delta} \\ &\quad + \left((A_l^\varepsilon - \bar{A}(\boldsymbol{x}_l))\nabla V_l, (1 - \rho^\varepsilon)\nabla\theta_l^\varepsilon\right)_{I_\delta}. \end{aligned}$$

364   Let $\mathcal{A} \in [L^\infty(\Omega; H^1(Y))]^{d\times d}$ be the solution of

365
$$\begin{cases} -\Delta_{\boldsymbol{y}}\mathcal{A} = A - \bar{A} & \text{in } Y, \\ \mathcal{A}(\boldsymbol{x}, \cdot) \text{ is } Y\text{-periodic} & \langle\mathcal{A}\rangle_Y = 0. \end{cases}$$

366 Since $\langle A(\boldsymbol{x}, \cdot) - \bar{A}(\boldsymbol{x})\rangle_Y = 0$, there exists a unique solution $\mathcal{A}$ such that

367           $$\| \mathcal{A} \|_{L^\infty(\Omega; (H^2(Y))} \le C\| A - \bar{A} \|_{L^\infty(\Omega; L^2(Y))} \le C.$$

368 Define $\mathcal{A}_l^\varepsilon := \mathcal{A}(\boldsymbol{x}_l, \cdot/\varepsilon)$ and it satisfies

369              $$-\varepsilon^2\Delta\mathcal{A}_l^\varepsilon = -\Delta_{\boldsymbol{y}}\mathcal{A}(\boldsymbol{x}_l, \cdot/\varepsilon) = A_l^\varepsilon - \bar{A}(\boldsymbol{x}_l).$$

370 Integration by part, we write

371       $$-\varepsilon^2(\Delta\mathcal{A}_l^\varepsilon\nabla V_l, \rho^\varepsilon\nabla\theta_l^\varepsilon)_{I_\delta} = \varepsilon^2(\nabla(\mathcal{A}_l^\varepsilon\nabla V_l), \nabla\theta_l^\varepsilon \otimes \nabla\rho^\varepsilon + \rho^\varepsilon\nabla^2\theta_l^\varepsilon)_{I_\delta}.$$

372 Using the trace inequality (3.15) with $z = \nabla\theta_l^\varepsilon$ and $\langle z \rangle_{I_\delta} = 0$, we bound the first term
373 in (3.17) as

374    $$|-\varepsilon^2(\Delta\mathcal{A}_l^\varepsilon\nabla V_l, \rho^\varepsilon\nabla\theta_l^\varepsilon)_{I_\delta}| \le C|\nabla V_l|\Big(\| \nabla_{\boldsymbol{y}}\mathcal{A}(\boldsymbol{x}_l, \cdot/\varepsilon) \|_{L^2(I_\delta^\xi)}\| \nabla\theta_l^\varepsilon \|_{L^2(I_\delta^\xi)}$$

375                                 $$+ \varepsilon\| \nabla_{\boldsymbol{y}}\mathcal{A}(\boldsymbol{x}_l, \cdot/\varepsilon) \|_{L^2(I_\delta)}\| \nabla^2\theta_l^\varepsilon \|_{L^2(I_\delta)}\Big)$$

376
377                                 $$\le C\varepsilon\| \nabla V_l \|_{L^2(I_\delta)}\| \nabla^2\theta_l^\varepsilon \|_{L^2(I_\delta)}.$$

378   Proceeding along the same line, we bound the second term in (3.17) as

379        $$|-\varepsilon^2(\Delta\mathcal{A}_l^\varepsilon\nabla V_l, (1-\rho^\varepsilon)\nabla\theta_l^\varepsilon)_{I_\delta}| \leq C|\nabla V_l|\|\Delta_{\boldsymbol{y}}\mathcal{A}\|_{L^2(I_\delta^\varepsilon)}\|\nabla\theta_l^\varepsilon\|_{L^2(I_\delta^\varepsilon)}$$

380        $$\leq C\varepsilon\|\nabla V_l\|_{L^2(I_\delta)}\|\nabla^2\theta_l^\varepsilon\|_{L^2(I_\delta)}.$$
381

382   Substituting the above two inequalities into (3.17), and using $\iota\|\nabla^2\theta_l^\varepsilon\|_{L^2(I_\delta)} \leq \|\theta_l^\varepsilon\|_\iota$,
383   we obtain (3.16).   □

384   We are ready to estimate $\langle A_l^\varepsilon\nabla\theta_l^\varepsilon\rangle_{I_\delta}$ with a dual argument.

385   LEMMA 3.6. *If* $\iota = \mu\varepsilon^\gamma$ *with* $0 < \gamma < 1$, *then*

386   (3.18) $$|\langle A_l^\varepsilon\nabla\theta_l^\varepsilon\rangle_{I_\delta}| \leq C\varepsilon^{2(1-\gamma)}|\nabla V_l|.$$

387   *Proof.* For any constant vector ∇W_l, we get

388        $$|I_\delta|\nabla W_l \cdot \langle A_l^\varepsilon\nabla\theta_l^\varepsilon\rangle_{I_\delta} = a_l^\varepsilon(\theta_l^\varepsilon, W_l) = a_l^\varepsilon(\theta_l^\varepsilon, W_l - w_l^\varepsilon) \leq \|\theta_l^\varepsilon\|_\iota\|W_l - w_l^\varepsilon\|_\iota$$

389        $$\leq C\varepsilon^{2(1-\gamma)}\|\nabla V_l\|_{L^2(I_\delta)}\|\nabla W_l\|_{L^2(I_\delta)},$$
390

391   where we have used (3.14) for w_l^ε with $z = \theta_l^\varepsilon$ in the second step, and (3.16) for both
392   v_l^ε and w_l^ε in the last step. This gives (3.18).   □

393   *Remark* 3.7. It is worth mentioning that the estimate (3.18) is independent of
394   δ and h when $\iota = \mu\varepsilon^\gamma$ with $0 < \gamma < 1$, which stands in striking contrast to the
395   corresponding estimate for the second-order homogenization problem; cf. [15, 11].

396   **3.3. Estimate of the corrector when** $\gamma \geq 1$. It is clear that V_l^ε satisfies

397   (3.19) $$a_l^\varepsilon(V_l^\varepsilon, z) = 0 \quad \text{for any } z \in H_0^2(I_\delta).$$

398   The following estimates for V_l^ε hang on the a priori estimate (3.3).

399   LEMMA 3.8. *If* $\iota = \mu\varepsilon^\gamma$ *with* $\gamma \geq 1$, *then*

400   (3.20) $$\|(1-\rho^\varepsilon)(V_l^\varepsilon - V_l)\|_\iota \leq C\sqrt{\varepsilon/\delta}\|\nabla V_l\|_{L^2(I_\delta)},$$

401   *and*

402   (3.21) $$\|\nabla V_l^\varepsilon\|_{L^2(I_\delta^\varepsilon)} + \iota\|\nabla^2 V_l^\varepsilon\|_{L^2(I_\delta^\varepsilon)} + \iota^2\|\nabla^3 V_l^\varepsilon\|_{L^2(I_\delta^\varepsilon)} \leq C\sqrt{\varepsilon/\delta}\|\nabla V_l\|_{L^2(I_\delta)}.$$

403   *Proof.* A direct calculation gives

404    $$\nabla\Big((1-\rho^\varepsilon)(V_l^\varepsilon - V_l)\Big) = \nabla_{\boldsymbol{y}}^\top\boldsymbol{\chi}^\gamma(\boldsymbol{x}_l, \cdot/\varepsilon)\nabla V_l(1-\rho^\varepsilon) - \varepsilon\boldsymbol{\chi}^\gamma(\boldsymbol{x}_l, \cdot/\varepsilon) \cdot \nabla V_l\nabla\rho^\varepsilon,$$

405   and

406    $$\varepsilon\nabla^2\Big((1-\rho^\varepsilon)(V_l^\varepsilon - V_l)\Big) = \nabla_{\boldsymbol{y}}^2(\boldsymbol{\chi}^\gamma \cdot \nabla V_l)(\boldsymbol{x}_l, \cdot/\varepsilon)(1-\rho^\varepsilon)$$

407    $$- \varepsilon\nabla_{\boldsymbol{y}}^\top\boldsymbol{\chi}^\gamma(\boldsymbol{x}_l, \cdot/\varepsilon)\nabla V_l \otimes \nabla\rho^\varepsilon - \varepsilon\nabla\rho^\varepsilon \otimes \nabla_{\boldsymbol{y}}^\top\boldsymbol{\chi}^\gamma(\boldsymbol{x}_l, \cdot/\varepsilon)\nabla V_l$$

408    $$+ \varepsilon^2\boldsymbol{\chi}(\boldsymbol{x}_l, \cdot/\varepsilon) \cdot \nabla V_l\nabla^2\rho^\varepsilon,$$
409

410   which together with (3.3) and (3.10) leads to

411    $$\|(1-\rho^\varepsilon)(V_l^\varepsilon - V_l)\|_\iota \leq C|\nabla V_l|\|(|\boldsymbol{\chi}| + |\nabla_{\boldsymbol{y}}\boldsymbol{\chi}| + \varepsilon^{\gamma-1}|\nabla_{\boldsymbol{y}}^2\boldsymbol{\chi}|)(\boldsymbol{x}_l, \cdot/\varepsilon)\|_{L^2(I_\delta^\varepsilon)}$$

412    $$\leq C\sqrt{\varepsilon\delta^{d-1}}|\nabla V_l|$$

413    $$\leq C\sqrt{\varepsilon/\delta}\|\nabla V_l\|_{L^2(I_\delta)}.$$
414

415   This gives (3.20).
416   The proof for (3.21) may be proceeded in the same way. We omit the details.   □

417   The next lemma concerns the error caused by the cell discretization.

418   LEMMA 3.9. *If $\iota = \mu\varepsilon^\gamma$ with $\gamma = 1$, or $\gamma > 1$ and $\|\nabla_{\boldsymbol{y}} A\|_{L^\infty(\Omega \times Y)}$ is bounded,*
419   *then*

420   (3.22) $\| (I - I_h)[\rho^\varepsilon(V_l^\varepsilon - V_l)] \|_\iota \le C \dfrac{h}{\varepsilon} \| \nabla V_l \|_{L^2(I_\delta)}.$

421   *Proof.* For $k = 2, 3$, a direct calculation gives

422            $\varepsilon^{k-1}|\nabla^k[\rho^\varepsilon(V_l^\varepsilon - V_l)]| = \sum_{j=0}^{k} \varepsilon^j |\nabla^j \rho^\varepsilon||\nabla_{\boldsymbol{y}}^{k-j}\boldsymbol{\chi}^\gamma(\boldsymbol{x}_l, \cdot/\varepsilon)||\nabla V_l|$

423                            $\le C|\nabla V_l| \sum_{j=0}^{k} |\nabla_{\boldsymbol{y}}^{k-j}\boldsymbol{\chi}^\gamma(\boldsymbol{x}_l, \cdot/\varepsilon)|.$
424

425   Using (3.3) when $\gamma = 1$ and using (3.5) when $\gamma > 1$ and $\|\nabla_{\boldsymbol{y}} A\|_{L^\infty(\Omega \times Y)}$ is bounded,
426   we obtain

427        $\| \nabla[\rho^\varepsilon(V_l^\varepsilon - V_l)] \|_\iota \le C|\nabla V_l|\Big(\varepsilon^{-1} \sum_{j=0}^{2} \| \nabla_{\boldsymbol{y}}^j \boldsymbol{\chi}^\gamma(\boldsymbol{x}_l, \cdot/\varepsilon) \|_{L^2(I_\delta)}$

428                            $+ \varepsilon^{\gamma-2} \sum_{j=0}^{3} \| \nabla_{\boldsymbol{y}}^j \boldsymbol{\chi}^\gamma(\boldsymbol{x}_l, \cdot/\varepsilon) \|_{L^2(I_\delta)}\Big)$

429
430                        $\le C\varepsilon^{-1}\| \nabla V_l \|_{L^2(I_\delta)}.$

431   For any $z \in H^3(I_\delta)$, it follows from the interpolation error estimate (2.4) that

432                        $\| (I - I_h)z \|_\iota \le Ch\| \nabla z \|_\iota.$

433   Choosing $z = \rho^\varepsilon(V_l^\varepsilon - V_l)$ and combining the above two inequalities, we obtain (3.22). □

434   *Remark 3.10.* If $\boldsymbol{\chi} \in [L^\infty(\Omega; H^3(Y))]^d$ holds, then the estimate (3.5) may be
435   improved to $\|\boldsymbol{\chi}^\gamma\|_{L^\infty(\Omega; H^3(Y))} \le C$, and the interpolation error (3.22) changes to
436    $\mathcal{O}(h^2/\varepsilon^2)$ when $\gamma \to \infty$. However, this would require extra smoothness assumption
437   on A.

438   We are ready to prove the estimate of the corrector.

439   LEMMA 3.11. *If $\iota = \mu\varepsilon^\gamma$ with $\gamma = 1$, or $\gamma > 1$ and $\|\nabla_{\boldsymbol{y}} A\|_{L^\infty(\Omega \times Y)}$ is bounded,*
440   *then*

441   (3.23) $\| \theta_l^\varepsilon \|_\iota \le C(\sqrt{\varepsilon/\delta} + h/\varepsilon)\| \nabla V_l \|_{L^2(I_\delta)}.$

442   *Proof.* A direct calculation gives

443            $a_l^\varepsilon(\theta_l^\varepsilon, \theta_l^\varepsilon) = a_l^\varepsilon(V_l^\varepsilon, \theta_l^\varepsilon) - a_l^\varepsilon(v_l^\varepsilon, \theta_l^\varepsilon)$
444                    $= \iota^2(\nabla^2 V_l^\varepsilon, \nabla^2 \theta_l^\varepsilon)_{I_\delta} + (A_l^\varepsilon \nabla V_l^\varepsilon - \bar{A}^\gamma(\boldsymbol{x}_l)\nabla V_l, \nabla \theta_l^\varepsilon)_{I_\delta}$
445                    $+ (\bar{A}^\gamma(\boldsymbol{x}_l)\nabla V_l, \nabla \theta_l^\varepsilon)_{I_\delta} - a_l^\varepsilon(v_l^\varepsilon, V_l^\varepsilon) + a_l^\varepsilon(v_l^\varepsilon, v_l^\varepsilon).$
446

447   Using the Hill's condition (2.10) and the definition (3.1) for v_l^ε, we substitute
448    $\langle \nabla \theta_l^\varepsilon \rangle_{I_\delta} = \langle \nabla(V_l^\varepsilon - V_l) \rangle_{I_\delta}$ into the third term, and employ $a_l^\varepsilon(v_l^\varepsilon, v_l^\varepsilon) = a_l^\varepsilon(v_l^\varepsilon, V_l)$ in

the last term. Then the above identity is reshaped into

$$
\begin{aligned}
a_l^\varepsilon(\theta_l^\varepsilon, \theta_l^\varepsilon) = {}& \iota^2(\nabla^2 V_l^\varepsilon, \nabla^2 \theta_l^\varepsilon)_{I_\delta} + \iota^2(\nabla \Delta V_l^\varepsilon, \nabla \theta_l^\varepsilon)_{I_\delta} \\
& + (A_l^\varepsilon \nabla V_l^\varepsilon - \bar{A}^\gamma(\boldsymbol{x}_l)\nabla V_l - \iota^2 \nabla \Delta V_l^\varepsilon, \nabla \theta_l^\varepsilon)_{I_\delta} \\
& + (\bar{A}^\gamma(\boldsymbol{x}_l)\nabla V_l, (\nabla V_l^\varepsilon - V_l))_{I_\delta} - a_l^\varepsilon(V_l^\varepsilon, V_l^\varepsilon - V_l) \\
& + a_l^\varepsilon(\theta_l^\varepsilon, V_l^\varepsilon - V_l).
\end{aligned}
\tag{3.24}
$$

Integration by parts, we obtain

$$
(\nabla \Delta V_l^\varepsilon, \rho^\varepsilon \nabla \theta_l^\varepsilon)_{I_\delta} = -(\rho^\varepsilon \nabla^2 V_l^\varepsilon, \nabla^2 \theta_l^\varepsilon)_{I_\delta} - (\nabla \rho^\varepsilon, \nabla^2 V_l^\varepsilon \nabla \theta_l^\varepsilon)_{I_\delta},
$$

from which we write the first line in (3.24) as

$$
\begin{aligned}
\iota^2(\nabla^2 V_l^\varepsilon, \nabla^2 \theta_l^\varepsilon)_{I_\delta} & + \iota^2(\nabla \Delta V_l^\varepsilon, \nabla \theta_l^\varepsilon)_{I_\delta} \\
& = \iota^2(\nabla^2 V_l^\varepsilon, \nabla^2 \theta_l^\varepsilon)_{I_\delta} + \iota^2(\nabla \Delta V_l^\varepsilon, \rho^\varepsilon \nabla \theta_l^\varepsilon)_{I_\delta} + \iota^2((1 - \rho^\varepsilon)\nabla \Delta V_l^\varepsilon, \nabla \theta_l^\varepsilon)_{I_\delta} \\
& = \iota^2((1 - \rho^\varepsilon)\nabla^2 V_l^\varepsilon, \nabla^2 \theta_l^\varepsilon)_{I_\delta} - \iota^2(\nabla \rho^\varepsilon, \nabla^2 V_l^\varepsilon \nabla \theta_l^\varepsilon)_{I_\delta} + \iota^2((1 - \rho^\varepsilon)\nabla \Delta V_l^\varepsilon, \nabla \theta_l^\varepsilon)_{I_\delta} \\
& \leq C\iota \| \theta_l^\varepsilon \|_\iota (\| \nabla^2 V_l^\varepsilon \|_{L^2(I_\delta^\varepsilon)} + \iota \| \nabla^3 V_l^\varepsilon \|_{L^2(I_\delta^\varepsilon)}) \\
& \leq \frac{1}{4}\| \theta_l^\varepsilon \|_\iota^2 + C\frac{\varepsilon}{\delta}\| \nabla V_l \|_{L^2(I_\delta)}^2,
\end{aligned}
$$

where we have used the Young's inequality and (3.21) in the last step.

For $i, j = 1, \cdots, d$, define the tensor

$$
B_{ij}(\boldsymbol{x}, \boldsymbol{y}) = (A_{ij} + A_{ik}\partial_{\boldsymbol{y}_k}\chi_j^\gamma - \mu^2 \varepsilon^{2(\gamma-1)}\partial_{\boldsymbol{y}_{ikk}}\chi_j^\gamma - \bar{A}_{ij}^\gamma)(\boldsymbol{x}, \boldsymbol{y}).
$$

By the definitions (3.2) and (3.7),

$$
\partial_{\boldsymbol{y}_i} B_{ij} = 0, \quad \langle B_{ij}\rangle_Y = 0.
$$

By [33, Theorem 3.1.1], there exists an anti-symmetric tensor

$$
\mathcal{B} \in [L^\infty(\Omega; H^1(Y))]^{d \times d \times d}
$$

such that

$$
\mathcal{B}_{kij} = -\mathcal{B}_{ikj}, \qquad \partial_{\boldsymbol{y}_k}\mathcal{B}_{kij} = B_{ij},
$$

and by (3.3),

$$
\begin{aligned}
\| \mathcal{B} \|_{L^\infty(\Omega; H^1(Y))} & \leq C\| B \|_{L^\infty(\Omega; L^2(Y))} \\
& \leq C\left(1 + \| \nabla_{\boldsymbol{y}}\boldsymbol{\chi}^\gamma \|_{L^\infty(\Omega; L^2(Y))} + \varepsilon^{2(\gamma-1)}\| \nabla_{\boldsymbol{y}}^3 \boldsymbol{\chi}^\gamma \|_{L^\infty(\Omega; L^2(Y))}\right) \\
& \leq C.
\end{aligned}
$$

Define $B_l^\varepsilon := B(\boldsymbol{x}_l, \cdot/\varepsilon)$, and $\mathcal{B}_l^\varepsilon := \mathcal{B}(\boldsymbol{x}_l, \cdot/\varepsilon)$, we obtain

$$
A_l^\varepsilon \nabla V_l^\varepsilon - \bar{A}^\gamma(\boldsymbol{x}_l)\nabla V_l - \iota^2 \nabla \Delta V_l^\varepsilon = B_l^\varepsilon \nabla V_l,
$$

and using the anti-symmetry of $\mathcal{B}$, we write

$$
\begin{aligned}
\langle \nabla \theta_l^\varepsilon \cdot \rho^\varepsilon B_l^\varepsilon \nabla V_l\rangle_{I_\delta} & = \varepsilon\langle \rho^\varepsilon \partial_i \theta_l^\varepsilon \partial_k \mathcal{B}_{l,kij}^\varepsilon \partial_j V_l\rangle_{I_\delta} \\
& = \varepsilon\langle \partial_i \theta_l^\varepsilon \partial_k(\rho^\varepsilon \mathcal{B}_{l,kij}^\varepsilon)\partial_j V_l\rangle_{I_\delta} - \varepsilon\langle \partial_i \theta_l^\varepsilon \mathcal{B}_{l,kij}^\varepsilon \partial_k \rho^\varepsilon \partial_j V_l\rangle_{I_\delta} \\
& = \varepsilon\langle \partial_i \theta_l^\varepsilon \mathcal{B}_{l,ikj}^\varepsilon \partial_k \rho^\varepsilon \partial_j V_l\rangle_{I_\delta} - \varepsilon\langle \rho^\varepsilon \partial_{ik}\theta_l^\varepsilon \mathcal{B}_{l,kij}^\varepsilon \partial_j V_l\rangle_{I_\delta} \\
& = \varepsilon\langle \nabla \theta_l^\varepsilon \cdot \mathcal{B}_l^\varepsilon(\nabla \rho^\varepsilon \otimes \nabla V_l)\rangle_{I_\delta}.
\end{aligned}
$$

Therefore, the second line of (3.24) is bounded by

$$(B_l^\varepsilon \nabla V_l, \nabla \theta_l^\varepsilon)_{I_\delta} = ((1 - \rho^\varepsilon) B_l^\varepsilon \nabla V_l, \nabla \theta_l^\varepsilon)_{I_\delta} + \varepsilon(\mathcal{B}_l^\varepsilon (\nabla \rho^\varepsilon \otimes \nabla V_l), \nabla \theta_l^\varepsilon)_{I_\delta}$$

$$\leq C |\nabla V_l| \| \nabla \theta_l^\varepsilon \|_{L^2(I_\delta)} (\| B_l^\varepsilon \|_{L^2(I_\delta^\varepsilon)} + \| \mathcal{B}_l^\varepsilon \|_{L^2(I_\delta^\varepsilon)})$$

$$\leq \frac{1}{4} \| \theta_l^\varepsilon \|_\iota^2 + C \frac{\varepsilon}{\delta} \| \nabla V_l \|_{L^2(I_\delta)}^2.$$

Next, we turn to the third line of (3.24). Since $V_l^\varepsilon - V_l$ is periodic, it is straightforward to verify

$$(A_l^\varepsilon \nabla V_l^\varepsilon, \nabla(V_l^\varepsilon - V_l))_{I_{\kappa\varepsilon}} + \iota^2 (\nabla^2 V_l^\varepsilon, \nabla^2 V_l^\varepsilon)_{I_{\kappa\varepsilon}} = 0$$

for the second term. Similarly, we use (3.9) for the first term. Using (3.21), we get

$$(\bar{A}^\gamma(\boldsymbol{x}_l) \nabla V_l, (\nabla V_l^\varepsilon - V_l))_{I_\delta} - a_l^\varepsilon(V_l^\varepsilon, V_l^\varepsilon - V_l)$$

$$\leq C \left(\| \nabla V_l \|_{L^2(I_\delta^\varepsilon)}^2 + \| \nabla V_l^\varepsilon \|_{L^2(I_\delta^\varepsilon)}^2 + \iota^2 \| \nabla^2 V_l^\varepsilon \|_{L^2(I_\delta^\varepsilon)}^2 \right)$$

$$\leq C \frac{\varepsilon}{\delta} \| \nabla V_l \|_{L^2(I_\delta)}^2.$$

Finally, we estimate the last line in (3.24). Choosing

$$z = (V_l^\varepsilon - V_l) - (V_l^\varepsilon - V_l)(1 - \rho^\varepsilon) - (I - I_h)[\rho^\varepsilon(V_l^\varepsilon - V_l)] \in V_h \cap H_0^2(I_\delta)$$

in (3.1) and (3.19), we obtain

$$a_l^\varepsilon(\theta_l^\varepsilon, V_l^\varepsilon - V_l) = a_l^\varepsilon(\theta_l^\varepsilon, (V_l^\varepsilon - V_l)(1 - \rho^\varepsilon) + (I - I_h)[\rho^\varepsilon(V_l^\varepsilon - V_l)])$$

$$\leq C \| \theta_l^\varepsilon \|_\iota \left(\| (V_l^\varepsilon - V)(1 - \rho^\varepsilon) \|_\iota + \| (I - I_h)[\rho^\varepsilon(V_l^\varepsilon - V_l)] \|_\iota \right)$$

$$\leq \frac{1}{4} \| \theta_l^\varepsilon \|_\iota^2 + C \left(\frac{\varepsilon}{\delta} + \frac{h^2}{\varepsilon^2} \right) \| \nabla V_l \|_{L^2(I_\delta)}^2,$$

where we have used (3.20) and (3.22) in the last step.

Substituting all the above inequalities into (3.24), we obtain (3.23). □

Remark 3.12. When $\gamma > 1$, the sequence A^ε H-converges to the second-order homogenization limit as $\varepsilon \to 0$. However, as shown in [27, 21], boundary layer degenerates the convergence rate of the discretization for general singular perturbations. This implies that for $\gamma \to \infty$, $\| v^\varepsilon - I_h v^\varepsilon \|_\iota$ is only $\mathcal{O}(\sqrt{h/\varepsilon})$, without any smoothness assumption on v^ε; see Appendix A. While the discretization error of the corrector given by (3.23) is $\mathcal{O}(h/\varepsilon)$.

We are ready to estimate $\langle A_l^\varepsilon \nabla \theta_l^\varepsilon \rangle_{I_\delta}$.

LEMMA 3.13. *If* $\iota = \mu \varepsilon^\gamma$ *with* $\gamma = 1$, *or* $\gamma > 1$ *and* $\| \nabla_{\boldsymbol{y}} A \|_{L^\infty(\Omega \times Y)}$ *is bounded, then*

(3.25)
$$|\langle A_l^\varepsilon \nabla \theta_l^\varepsilon \rangle_{I_\delta}| \leq C \left(\frac{\varepsilon}{\delta} + \frac{h^2}{\varepsilon^2} \right) |\nabla V_l|.$$

Proof. For any constant vector ∇W_l, using the fact that the homogenized effective matrix for the dual problem of (1.1) is $\bar{A}^\top$ as shown in [33, Lemma 2.2.5], we write

(3.26)
$$|I_\delta| \nabla W_l \cdot \langle A_l^\varepsilon \nabla \theta_l^\varepsilon \rangle_{I_\delta} = a_l^\varepsilon(\theta_l^\varepsilon, W_l) = a_l^\varepsilon(\theta_l^\varepsilon, W_l^\varepsilon) - a_l^\varepsilon(\theta_l^\varepsilon, W_l^\varepsilon - W_l)$$
$$= \iota^2 (\nabla^2 \theta_l^\varepsilon, \nabla^2 W_l^\varepsilon)_{I_\delta} + (\nabla \theta_l^\varepsilon, (A_l^\varepsilon)^\top \nabla W_l^\varepsilon - (\bar{A}^\gamma)^\top(\boldsymbol{x}_l) \nabla W_l)$$
$$+ (\bar{A}^\gamma(\boldsymbol{x}_l) \nabla \theta_l^\varepsilon, \nabla W_l)_{I_\delta} - a_l^\varepsilon(\theta_l^\varepsilon, W_l^\varepsilon - W_l).$$

Proceeding along the same line that leads to the first line and the second line in (3.24), we obtain

$$\iota^2(\nabla^2\theta_l^\varepsilon, \nabla^2 W_l^\varepsilon)_{I_\delta} + (\nabla\theta_l^\varepsilon, (A_l^\varepsilon)^\top \nabla W_l^\varepsilon - (\bar{A}^\gamma)^\top(\boldsymbol{x}_l)\nabla W_l)$$

$$\le C\sqrt{\varepsilon/\delta}\|\,\theta_l^\varepsilon\,\|_\iota\|\,\nabla W_l\,\|_{L^2(I_\delta)}$$

$$\le C\Big(\frac{\varepsilon}{\delta} + \frac{h^2}{\varepsilon^2}\Big)\|\,\nabla V_l\,\|_{L^2(I_\delta)}\|\,\nabla W_l\,\|_{L^2(I_\delta)},$$

where we have used (3.23) in the last step.

By (2.10) and (3.9), using (3.21), we bound the second to last term in (3.26) by

$$(\bar{A}^\gamma(\boldsymbol{x}_l)\nabla\theta_l^\varepsilon, \nabla W_l)_{I_\delta} = (\bar{A}^\gamma(\boldsymbol{x}_l)\nabla(V_l^\varepsilon - V_l), \nabla W_l)_{I_\delta}$$

$$\le C\|\,\nabla(V_l^\varepsilon - V_l)\,\|_{L^2(I_\delta^\varepsilon)}\|\,\nabla W_l\,\|_{L^2(I_\delta^\varepsilon)}$$

$$\le C\frac{\varepsilon}{\delta}\|\,\nabla V_l\,\|_{L^2(I_\delta)}\|\,\nabla W_l\,\|_{L^2(I_\delta)}.$$

Choosing

$$z = W_l^\varepsilon - W_l - (W_l^\varepsilon - W_l)(1 - \rho^\varepsilon) - (I - I_h)[\rho^\varepsilon(W_l^\varepsilon - W_l)] \in V_h \cap H_0^2(I_\delta)$$

in (3.1) and (3.19), we estimate the last term in (3.26) as

$$-a_l^\varepsilon(\theta_l^\varepsilon, W_l^\varepsilon - W_l) = -a_l^\varepsilon(\theta_l^\varepsilon, (W_l^\varepsilon - W_l)(1 - \rho^\varepsilon) + (I - I_h)[\rho^\varepsilon(W_l^\varepsilon - W_l)])$$

$$\le C\|\,\theta_l^\varepsilon\,\|_\iota\Big(\|\,(W_l^\varepsilon - W_l)(1 - \rho^\varepsilon)\,\|_\iota + \|\,(I - I_h)[\rho^\varepsilon(W_l^\varepsilon - W_l)]\,\|_\iota\Big)$$

$$\le C\Big(\frac{\varepsilon}{\delta} + \frac{h^2}{\varepsilon^2}\Big)\|\,\nabla V_l\,\|_{L^2(I_\delta)}\|\,\nabla W_l\,\|_{L^2(I_\delta)},$$

where we have used an analogy of (3.20) and (3.22) for W_l^ε and (3.23) in the last step.

Substituting the above inequalities into (3.26), we obtain

$$|\nabla W_l \cdot \langle A_l^\varepsilon \nabla \theta_l^\varepsilon \rangle_{I_\delta}| \le \frac{C}{|I_\delta|}\left(\frac{\varepsilon}{\delta} + \frac{h^2}{\varepsilon^2}\right)\|\,\nabla V_l\,\|_{L^2(I_\delta)}\|\,\nabla W_l\,\|_{L^2(I_\delta)}$$

$$\le C\left(\frac{\varepsilon}{\delta} + \frac{h^2}{\varepsilon^2}\right)|\nabla V_l||\nabla W_l|.$$

This leads to (3.25).   □

Summing up all the estimates in this part, we are ready to bound $e(\mathrm{HMM})$.

THEOREM 3.14. *If $\iota = \mu\varepsilon^\gamma$ with $\gamma > 0$ and $A \in [C^{0,1}(\Omega; L^\infty(\mathbb{R}^d))]^{d\times d}$, then*

$$(3.27)\quad e(\mathrm{HMM}) \le C \begin{cases} \delta + \varepsilon/\delta + \varepsilon^{2(1-\gamma)} & 0 < \gamma < 1, \\ \delta + \varepsilon/\delta + h^2/\varepsilon^2 & \gamma = 1, \\ \delta + \varepsilon/\delta + \varepsilon^{2(\gamma-1)} + h^2/\varepsilon^2 & \gamma > 1, \|\,\nabla_{\boldsymbol{y}}A\,\|_{L^\infty(\Omega\times Y)} < \infty. \end{cases}$$

Moreover, if A^ε is periodic, i.e., $A(\boldsymbol{x}, \boldsymbol{y})$ is independent of $\boldsymbol{x}$, and δ is an integer multiple of ε, then

$$(3.28)\qquad e(\mathrm{HMM}) \le C \begin{cases} \varepsilon^{2(1-\gamma)} & 0 < \gamma < 1, \\ \varepsilon/\delta + h^2/\varepsilon^2 & \gamma = 1, \\ \varepsilon/\delta + \varepsilon^{2(\gamma-1)} + h^2/\varepsilon^2 & \gamma > 1, \|\,\nabla_{\boldsymbol{y}}A\,\|_{L^\infty(\Omega\times Y)} < \infty. \end{cases}$$

Proof. Substituting (3.8) and Lemma 3.2, Lemma 3.3, Lemma 3.6, Lemma 3.13 into (3.11), we obtain (3.27).

If $A_l^\varepsilon = A^\varepsilon$ and δ/ε is a positive integer, then error bounds in Lemma 3.2 and Lemma 3.3 vanish, the estimate (3.27) changes to (3.28).   □

4. Numerical experiments. In this part, we report numerical examples to validate the accuracy of the proposed method. We shall employ the Specht element to solve the cell problems because it is one of the best thin plate triangles with 9 degrees of freedom that currently available [39, citation in p. 345]. Theorem 3.14 may be extended to the nonconforming microscale discretization with the aid of the enriching operator [8, 21]. We refer to [23] for more details.

We define a piecewise space as

$$H^m_{\mathcal{T}_h}(I_\delta) := \{ z_h \in L^2(I_\delta) \mid z_h|_K \in H^m(K) \quad \text{for all } K \in \mathcal{T}_h \},$$

which is equipped with the broken norm

$$\| z_h \|^2_{H^m_{\mathcal{T}_h}(I_\delta)} := \| z_h \|^2_{L^2(I_\delta)} + \sum_{i=1}^m \| \nabla^i_h z_h \|^2_{L^2(I_\delta)},$$

where

$$\| \nabla^i_h z_h \|^2_{L^2(I_\delta)} = \sum_{K \in \mathcal{T}_h} \| \nabla^i z_h \|^2_{L^2(K)}.$$

Let $\mathcal{V}^i_h$ be set of the interior vertices of $\mathcal{T}_h$, and $\mathcal{V}^b_h$ be the set of the vertices on the boundary, and $\mathcal{E}^i_h$ be the interior edges of $\mathcal{T}_h$. The Specht element introduced in [35] is an H^1-conforming but H^2-nonconforming element, which is defined as

$$X_h := \Big\{ z_h \in H^1(I_\delta) \mid z_h|_K \in P_K \text{ for all } K \in \mathcal{T}_h, $$

$$\nabla z_h(\boldsymbol{a}) \text{ are continuous for all } \boldsymbol{a} \in \mathcal{V}^i_h \Big\},$$

where $P_K \supset \mathbb{P}_2$ is the local space on K. According to [34, § 3.8],

$$\langle \llbracket \partial_{\boldsymbol{n}} z_h \rrbracket \rangle_e = 0 \qquad \text{for all} \quad z_h \in X_h, \quad e \in \mathcal{E}^i_h.$$

The interpolation operator (2.4) is defined in [21, Theorem 3].

The bilinear form a^ε in (2.5) is replaced by $a^\varepsilon_h : H^2_{\mathcal{T}_h}(I_\delta) \times H^2_{\mathcal{T}_h}(I_\delta) \to \mathbb{R}$ given by

$$a^\varepsilon_h(v_h, z_h) := (A^\varepsilon \nabla v_h, \nabla z_h)_{I_\delta} + \iota^2(\nabla^2_h v_h, \nabla^2_h z_h)_{I_\delta} \quad \text{for any } v_h, z_h \in H^2_{\mathcal{T}_h}(I_\delta),$$

and the four type boundary conditions are defined by

1. Essential boundary condition:

$$V_h = \{ z_h \in X_h \cap H^1_0(I_\delta) \mid \nabla z_h(\boldsymbol{a}) = \boldsymbol{0} \text{ for all } \boldsymbol{a} \in \mathcal{V}^b_h \};$$

2. Natural boundary condition: $V_h = X_h \cap H^1_0(I_\delta)$;
3. Free boundary condition: $V_h = \{ z_h \in X_h \cap L^2_0(I_\delta) \mid \langle \nabla z_h \rangle_{I_\delta} = \boldsymbol{0} \}$;
4. Periodic boundary condition: $V_h = X_h \cap L^2_0(I_\delta) \cap H^1_{\text{per}}(I_\delta)$.

The weighted norm over V_h is defined by

$$\| \cdot \|_{\iota,h} := \| \nabla \cdot \|_{L^2(I_\delta)} + \iota \| \nabla^2_h \cdot \|_{L^2(I_\delta)}.$$

Hence a^ε_h is bounded and covercive over V_h with respect to $\| \cdot \|_{\iota,h}$. By Lax-Milgram theorem, the above cell problem has a unique solution v^ε_h.

We select $V_l = x_1$ and x_2 in (2.5), and calculate the solution v^ε_1 and v^ε_2, respectively. The effect matrix is computed by (2.6). Motivated by [37, § 2.4], we use the weighted averaging methods to improve the accuracy:

$$A_H(\boldsymbol{x}_l) := \big(\langle \omega A^\varepsilon \nabla v^\varepsilon_1 \rangle_{I_\delta} \quad \langle \omega A^\varepsilon \nabla v^\varepsilon_2 \rangle_{I_\delta} \big) \big(\langle \omega \nabla v^\varepsilon_1 \rangle_{I_\delta} \quad \langle \omega \nabla v^\varepsilon_2 \rangle_{I_\delta} \big)^{-1},$$

where the weighted function

(4.1) $\omega(\boldsymbol{x}) = \Big(1 + \cos(2\pi(x_1 - x_{l,1})/\delta)\Big)\Big(1 + \cos(2\pi(x_2 - x_{l,2})/\delta)\Big).$
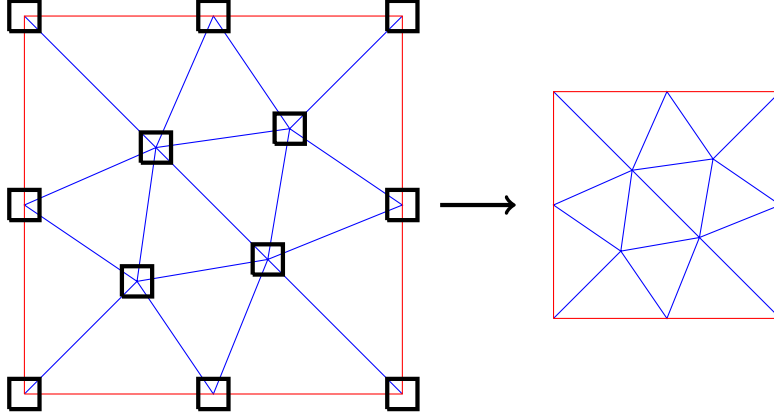

FIG. 1. *Left: macroscopic meshes $\mathcal{T}_H$ on domain Ω; Right: microscopic meshes $\mathcal{T}_h$ on the cell I_δ.*

Let $\Omega = (0,1)^2$ and $\mu = 1$, we divide Ω into non-uniform meshes, solve the cell problems on all vertices $\boldsymbol{a}$ in parallel, and $\omega_l = 1/3$; See Figure 1.

4.1. Accuracy of the effective matrix. In the first example, we test a layered material, i.e., the coefficient A^ε depends only on x_1,

$A^\varepsilon(\boldsymbol{x}) = \dfrac{1}{2\pi - \cos(2\pi x_1/\varepsilon)} \begin{pmatrix} 50 + 4\pi^2 \cos(2\pi x_1/\varepsilon) & (4\pi^2 + 25/\pi)\sin(2\pi x_1/\varepsilon) \\ 0 & 4\pi^2 - 1 + \sin(2\pi x_1/\varepsilon) \end{pmatrix}.$

A direct computation gives the analytical expression of the effective matrix

$$\bar{A} = \begin{pmatrix} \bar{A}_{11} & 0 \\ 0 & \sqrt{4\pi^2 - 1} \end{pmatrix}$$

with

$$\bar{A}_{11} = \begin{cases} (50 + 8\pi^3)/\sqrt{4\pi^2 - 1} - 4\pi^2 & 0 < \gamma < 1, \\ 25/\pi & \gamma = 1, \\ 4\pi^2/[(25 + 4\pi^3)/\sqrt{625 - 4\pi^4} - 1] & \gamma > 1, \end{cases}$$

We are interested in whether the resonance error and the discretization error are optimal. Hence, we solely contemplate on the scenario that A^ε is periodic, and do not pay attention to the error caused by the local periodicity.

We report the relative error

$$e_F(\text{HMM}) := \max_{\boldsymbol{a}\text{ the vertices of }\mathcal{T}_H} \frac{\|(A_H - \bar{A})(\boldsymbol{a})\|_F}{\|\bar{A}(\boldsymbol{a})\|_F},$$

where $\|\cdot\|_F$ represents the Frobenius norm of the matrix. Note that $e_F(\text{HMM})$ is equivalent to $e(\text{HMM})$, while it is simpler for implementation.

Without employing the weighted average, it follows from Table 1 that $e_F(\text{HMM})$ are $\mathcal{O}(1)$ when $\gamma < 1$ and $\mathcal{O}(\delta^{-1})$ when $\gamma \geq 1$, which are aligning perfectly with Theorem 3.14.

TABLE 1

e_F (HMM) w.r.t. the cell size δ for the 1st example, with and without weighted averaging, $\varepsilon = 2^{-5}$ and $h = 2^{-9}$.

$\gamma\backslash\delta$	2^{-5}	2^{-4}	2^{-3}	2^{-2}	2^{-1}
	Essential boundaries without weighted average				
0.25	9.8755e-05	2.3796e-04	3.1778e-04	3.5597e-04	3.7392e-04
rate		-1.27	-0.42	-0.16	-0.07
1.00	4.7228e-02	2.7798e-02	1.4240e-02	7.1454e-03	3.6163e-03
rate		0.76	0.97	0.99	0.98
4.00	2.3679e-01	1.1183e-01	5.0872e-02	2.3541e-02	1.1154e-02
rate		1.08	1.14	1.11	1.08
	Natural boundaries without weighted average				
0.25	3.6687e-01	4.1168e-04	4.9067e-04	5.1721e-04	5.0822e-04
rate		9.80	-0.25	-0.08	0.03
1.00	2.7187e-02	1.3792e-02	6.9795e-03	3.7269e-03	2.2220e-03
rate		0.98	0.98	0.91	0.75
4.00	2.2666e-01	1.0711e-01	4.8668e-02	2.2475e-02	1.0629e-02
rate		1.08	1.14	1.11	1.08
	Free boundaries without weighted average				
0.25	9.7224e-04	9.6194e-04	9.3580e-04	8.5284e-04	6.9166e-04
rate		0.02	0.04	0.13	0.30
1.00	5.3796e-02	3.0771e-02	1.5599e-02	7.7678e-03	3.8279e-03
rate		0.81	0.98	1.01	1.02
4.00	9.2755e-03	3.1555e-03	1.4309e-03	6.5085e-04	3.0399e-04
rate		1.56	1.14	1.14	1.10
	Essential boundaries with weighted average (4.1)				
0.25	3.6681e-01	3.7678e-04	3.8763e-04	3.8998e-04	3.9050e-04
rate		9.93	-0.04	-0.01	-0.00
1.00	4.2970e-01	4.1742e-03	8.3335e-04	2.4794e-04	1.7062e-04
rate		6.69	2.32	1.75	0.54
4.00	7.7975e-01	1.2809e-02	1.8316e-03	2.9735e-04	6.6755e-05
rate		5.93	2.81	2.62	2.16

We observe that the cell problems with free boundary conditions perform slightly better when $\gamma > 1$, which seems caused by the boundary layer effects. Moreover, the weighted average may lead to a remarkable reduction in errors for large γ.

TABLE 2

e_F (HMM) w.r.t. the multiscale ε for the 1st example, essential boundary conditions without weighted average, $\delta = 2^{-1}$ and $h = 2^{-9}$.

$\gamma\backslash\varepsilon$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
0.25	6.6896e-03	5.5478e-03	2.5850e-03	1.0149e-03	3.7392e-04
rate		0.27	1.10	1.35	1.44
0.50	9.2802e-03	1.0819e-02	7.1842e-03	4.0175e-03	2.1022e-03
rate		-0.22	0.59	0.84	0.93
0.75	1.2781e-02	2.0616e-02	1.9371e-02	1.5433e-02	1.1503e-02
rate		-0.69	0.09	0.33	0.42
1.00	4.5901e-02	2.7590e-02	1.4062e-02	7.0174e-03	3.6163e-03
rate		0.73	0.97	1.00	0.96
1.50	4.9019e-01	4.0570e-01	2.9837e-01	2.0229e-01	1.3040e-01
rate		0.27	0.44	0.56	0.63
2.00	4.6424e-01	2.8744e-01	1.3023e-01	5.2731e-02	2.0802e-02
rate		0.69	1.14	1.30	1.34
4.00	3.1211e-01	1.0780e-01	4.8341e-02	2.3041e-02	1.1154e-02
rate		1.53	1.16	1.07	1.05

Theorem 3.14 illustrates how the combined effect of the singular perturbation and the homogenization when $\iota \to 0$ in different regimes. It follows from Table 2, Table 3

TABLE 3

e_F(HMM) w.r.t. the multiscale ε for the 1st example, natural boundary conditions without weighted average, $\delta = 2^{-1}$ and $h = 2^{-9}$.

$\gamma\backslash\varepsilon$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
0.25	1.4923e-02	9.5660e-03	4.2174e-03	1.5986e-03	5.0822e-04
rate		0.64	1.18	1.40	1.65
0.50	2.0190e-02	1.7877e-02	1.0954e-02	5.7296e-03	2.5736e-03
rate		0.18	0.71	0.93	1.15
0.75	2.6924e-02	3.1982e-02	2.6782e-02	1.9528e-02	1.2810e-02
rate		-0.25	0.26	0.46	0.61
1.00	2.7033e-02	1.3621e-02	6.8198e-03	3.6600e-03	2.2220e-03
rate		0.99	1.00	0.90	0.72
1.50	4.4854e-01	3.6039e-01	2.6744e-01	1.8583e-01	1.2302e-01
rate		0.32	0.43	0.53	0.60
2.00	4.1122e-01	2.4308e-01	1.1438e-01	4.7150e-02	1.8190e-02
rate		0.76	1.09	1.28	1.37
4.00	2.7173e-01	1.0308e-01	4.7668e-02	2.2546e-02	1.0629e-02
rate		1.40	1.11	1.08	1.08

TABLE 4

e_F(HMM) w.r.t. the multiscale ε for the 1st example, free boundary conditions without weighted average, $\delta = 2^{-1}$ and $h = 2^{-9}$.

$\gamma\backslash\varepsilon$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
0.25	5.6414e-02	2.0473e-02	7.2856e-03	2.4598e-03	6.9166e-04
rate		1.46	1.49	1.57	1.83
0.50	7.2549e-02	3.4936e-02	1.6430e-02	7.3286e-03	2.9035e-03
rate		1.05	1.09	1.16	1.34
0.75	9.1062e-02	5.6303e-02	3.5530e-02	2.2386e-02	1.3497e-02
rate		0.69	0.66	0.67	0.73
1.00	5.7404e-02	3.1341e-02	1.7122e-02	9.1322e-03	3.8279e-03
rate		0.87	0.87	0.91	1.25
1.50	3.9104e-01	3.1895e-01	2.4111e-01	1.7063e-01	1.1453e-01
rate		0.29	0.40	0.50	0.58
2.00	3.1896e-01	1.7056e-01	7.2766e-02	2.5657e-02	7.7513e-03
rate		0.90	1.23	1.50	1.73
4.00	7.3885e-02	6.9724e-03	2.5177e-03	9.4015e-04	3.0399e-04
rate		3.41	1.47	1.42	1.63

TABLE 5

e_F(HMM) w.r.t. the multiscale ε for the 1st example, periodic boundary conditions without weighted average, $\delta = \varepsilon$ and $h = 2^{-8}\varepsilon$.

$\gamma\backslash\varepsilon$	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
0.25	3.9187e-04	1.3874e-04	4.9131e-05	1.8237e-05	6.4369e-06
rate		1.50	1.50	1.43	1.50
0.50	2.2037e-03	1.1058e-03	5.5410e-04	2.7795e-04	1.3983e-04
rate		0.99	1.00	1.00	0.99
0.75	1.2068e-02	8.6315e-03	6.1534e-03	4.3773e-03	3.1074e-03
rate		0.48	0.49	0.49	0.49
1.00	7.9881e-07	6.5756e-07	6.7266e-07	1.1896e-06	1.6055e-06
rate		0.28	-0.03	-0.82	-0.43
1.50	1.1418e-01	7.2736e-02	4.4160e-02	2.5597e-02	1.4225e-02
rate		0.65	0.72	0.79	0.85
2.00	7.6310e-03	2.0436e-03	5.2185e-04	1.3121e-04	3.2850e-05
rate		1.90	1.97	1.99	2.00
2.50	2.6191e-04	3.2850e-05	4.1084e-06	5.1389e-07	6.4574e-08
rate		3.00	3.00	3.00	2.99

616 and Table 4 that the convergence rates of e_F (HMM) are nearly $\mathcal{O}(\varepsilon^{2(1-\gamma)})$ when
617 $\gamma < 1$. Due to the influence of the resonance error, it reduces the convergence rate
618 when $\gamma > 1$.
619   To eliminate the effect of resonance error, we tested the periodic boundary con-
620 ditions as presented in Table tab:epsPeriodic. The table shows that e_F(HMM) are
621 $\mathcal{O}(\varepsilon^{2|1-\gamma|})$. When $\gamma = 1$, relative errors are small and the rates of convergence are
independent of ε.

TABLE 6
e_F(HMM) w.r.t. the microscopic mesh size h for the 1st example, with weighted average, $\varepsilon = 2^{-5}$ and $\delta = 2^{-1}$.

$\gamma \backslash h$	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
	Essential boundary condition				
0.25	1.6364e-04	3.0990e-04	3.7226e-04	3.8698e-04	3.9050e-04
rate		-0.92	-0.26	-0.06	-0.01
1.00	4.6743e-02	1.1214e-02	2.5144e-03	6.1601e-04	1.7062e-04
rate		2.06	2.16	2.03	1.85
4.00	3.4707e-01	7.4217e-02	6.0430e-03	3.5164e-04	6.6755e-05
rate		2.23	3.62	4.10	2.40
	Natural boundary condition				
0.25	1.6295e-04	3.0940e-04	3.7225e-04	3.8701e-04	3.9053e-04
rate		-0.93	-0.27	-0.06	-0.01
1.00	4.6750e-02	1.1218e-02	2.5145e-03	6.1755e-04	1.7228e-04
rate		2.06	2.16	2.03	1.84
4.00	3.4660e-01	7.4208e-02	6.0320e-03	3.4621e-04	6.4786e-05
rate		2.22	3.62	4.12	2.42
	Free boundary condition				
0.25	1.6314e-04	3.0936e-04	3.7236e-04	3.8710e-04	3.9060e-04
rate		-0.92	-0.27	-0.06	-0.01
1.00	4.6683e-02	1.1185e-02	2.5002e-03	6.0859e-04	1.6431e-04
rate		2.06	2.16	2.04	1.89
4.00	3.4655e-01	7.4172e-02	6.0132e-03	3.1711e-04	3.1241e-05
rate		2.22	3.62	4.25	3.34
	Periodic boundary condition				
0.25	1.6411e-04	3.1017e-04	3.7241e-04	3.8710e-04	3.9061e-04
rate		-0.92	-0.26	-0.06	-0.01
1.00	4.6695e-02	1.1188e-02	2.5004e-03	6.0227e-04	1.5809e-04
rate		2.06	2.16	2.05	1.93
4.00	3.4656e-01	7.4172e-02	6.0110e-03	3.1725e-04	2.0052e-05
rate		2.22	3.63	4.24	3.98

622
623   Finally, we employ the weighted average to mitigate the resonance errors, primar-
624 ily focusing on the error caused by the cell discretization. It followed from Table 6 that
625 e_F(HMM) are $\mathcal{O}(1)$ when $\gamma < 1$ and at least $\mathcal{O}(h^2)$ when $\gamma \geq 1$, which is consistent
626 with Theorem 3.14. Despite the stronger boundary layer effect, the performance of
627 the essential boundary conditions are comparable with the performance of the natural
628 boundary condition. It seems to achieve nearly $\mathcal{O}(h^4)$ when $\gamma > 1$ since the corrector
629 χ are smoother than we have expected; See Remarkrmk:smooth.

630   **4.2. Accuracy of the homogenized solution.** In the second example, we test
631 the accuracy of HMM for singular perturbation homogenization problems with the
632 locally periodic coefficient

633
$$A^\varepsilon(\boldsymbol{x}) = \begin{pmatrix} \dfrac{20\pi + 4\pi^2 \cos(2\pi x_1/\varepsilon)}{2\pi - \cos(2\pi x_1/\varepsilon)} & 2 + \sin(2\pi x_1) + \cos(2\pi x_2/\varepsilon) \\ 3 + \cos(2\pi x_2) + \sin(2\pi x_1/\varepsilon) & \dfrac{22\pi + 4\pi^2 \sin(2\pi x_2/\varepsilon)}{2\pi - \sin(2\pi x_2/\varepsilon)} \end{pmatrix}.$$

634   A direct calculation gives the effective matrix

$$\bar{A}(\boldsymbol{x}) = \begin{pmatrix} \bar{A}_{11} & 2 + \sin(2\pi x_1) \\ 3 + \cos(2\pi x_2) & \bar{A}_{22} \end{pmatrix}$$

636   with

$$\bar{A}_{11} = \begin{cases} 4\pi(2\pi^2 + 5)/\sqrt{4\pi^2 - 1} - 4\pi^2 & 0 < \gamma < 1, \\ 10 & \gamma = 1, \\ 4\pi^2/[(5 + 2\pi^2)/\sqrt{25 - \pi^2} - 1] & \gamma > 1, \end{cases}$$

638   and

$$\bar{A}_{22} = \begin{cases} 2\pi(4\pi^2 + 11)/\sqrt{4\pi^2 - 1} - 4\pi^2 & 0 < \gamma < 1, \\ 11 & \gamma = 1, \\ 4\pi^2/[(11 + 4\pi^2)/\sqrt{121 - 4\pi^2} - 1] & \gamma > 1. \end{cases}$$

640   Let the solution of the homogenization problem $\bar{u}(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2)$, and we
641 compute the source term f by (1.3).
642   We solve the microscopic cell problems (2.5) on cells posed over all vertices of $\mathcal{T}_H$
643 and the macroscopic problem (2.1) by the vertices based integration scheme [11, eq.
644 (2.7)] with $\boldsymbol{x}_l$ the vertices of K and $\omega_l = 1/3$ for $l = 1, 2, 3$. We report the relative
H^1-error $\|\nabla(\bar{u} - u_H)\|_{L^2(\Omega)}/\|\nabla\bar{u}\|_{L^2(\Omega)}$ in Table 7.

TABLE 7

Relative H^1-errors w.r.t. the macroscopic mesh size H for the 2nd example, without weighted average, $\varepsilon = 2^{-5}, \delta = 2^{-2}$ and $h = 2^{-8}$.

$\gamma \backslash H$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
	Essential boundary condition				
0.25	8.2439e-01	4.7977e-01	2.4203e-01	1.2392e-01	6.1370e-02
rate		0.78	0.99	0.97	1.01
1.00	8.2026e-01	4.7772e-01	2.4103e-01	1.2357e-01	6.1434e-02
rate		0.78	0.99	0.96	1.01
4.00	8.1222e-01	4.7459e-01	2.3968e-01	1.2412e-01	6.3624e-02
rate		0.78	0.99	0.95	0.96
	Natural boundary condition				
0.25	8.2447e-01	4.7982e-01	2.4206e-01	1.2393e-01	6.1377e-02
rate		0.78	0.99	0.97	1.01
1.00	8.2326e-01	4.7941e-01	2.4177e-01	1.2381e-01	6.1348e-02
rate		0.78	0.99	0.97	1.01
4.00	8.1382e-01	4.7545e-01	2.3992e-01	1.2390e-01	6.2933e-02
rate		0.78	0.99	0.95	0.98
	Free boundary condition				
0.25	8.2459e-01	4.7989e-01	2.4209e-01	1.2396e-01	6.1389e-02
rate		0.78	0.99	0.97	1.01
1.00	8.2586e-01	4.8089e-01	2.4256e-01	1.2423e-01	6.1639e-02
rate		0.78	0.99	0.97	1.01
4.00	8.2430e-01	4.8134e-01	2.4248e-01	1.2415e-01	6.1602e-02
rate		0.78	0.99	0.97	1.01
	Periodic boundary condition				
0.25	8.2441e-01	4.7978e-01	2.4204e-01	1.2393e-01	6.1372e-02
rate		0.78	0.99	0.97	1.01
1.00	8.2364e-01	4.7960e-01	2.4191e-01	1.2387e-01	6.1350e-02
rate		0.78	0.99	0.97	1.01
4.00	8.2340e-01	4.8078e-01	2.4222e-01	1.2403e-01	6.1508e-02
rate		0.78	0.99	0.97	1.01

645
646   Even though we do not employ the weighted average, it is accurate enough to
647 achieve the precision we desire. Meanwhile the first order rate of convergences of the
648 relative H^1-error has been achieved for $\|\bar{u} - u_H\|_{H^1}$, as predicted in (2.11).

5. Conclusion. We introduce a HMM-FEM for solving a singular perturbation homogenization problem. Our method is robust, as it does not rely on the relation between ι and ε, and explicit form of A^ε, thereby guaranteeing convergence. Unlike the classical second-order elliptic homogenization problem, our analysis reveals that, in certain scenarios, the resonance error and discretization error depend solely on ε. Additionally, we have observed boundary layer effects which, importantly, do not affect the convergence rate, even when essential boundary conditions are imposed on the cell problems.

Appendix A. Estimates for the solution of the cell problems. In this appendix, we discuss the discretization error of the cell problem, which is generally dominated by the interpolation error of the cell solution, which, unfortunately, may not be entirely reasonable. On the one hand, the regularity of the solution to Problem (2.3) depends on the smoothness of the domain. For general boundary value problems on convex polygonal domains, the solution $v^\varepsilon \notin H^3(I_\delta)$; see [7, Theorem 2]. Conversely, the cell is usually a cube. On the other hand, what exacerbates the situation is that, even if $v^\varepsilon \in H^3(I_\delta)$ can be guaranteed for essential boundary value problems when $d = 2, 3$, the interpolation error is only $\mathcal{O}(\sqrt{h/\varepsilon})$ as $\gamma \to \infty$ due to the boundary layer effect. We shall elaborate on this issue.

As $\varepsilon \to 0$, Problem (2.3) with the essential boundary condition tends to

(A.1)
$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla v_0^\varepsilon) = 0 & \text{in } I_\delta, \\ v_0^\varepsilon = V_l & \text{on } \partial I_\delta. \end{cases}$$

The following lemma regarding the symmetrical matrix A has been provided in [12, Appendix A], and we extend it to non-symmetrical matrix A.

LEMMA A.1. *If* $\| \nabla_{\boldsymbol{y}} A \|_{L^\infty(\Omega \times Y)}$ *is bounded, then*

(A.2)
$$\| \nabla v_0^\varepsilon \|_{L^2(I_\delta)} + \varepsilon \| \nabla^2 v_0^\varepsilon \|_{L^2(I_\delta)} \le C \| \nabla V_l \|_{L^2(I_\delta)}.$$

Proof. Multiplying both sides of (A.2) by $z = v_0^\varepsilon - V_l$, integration by parts and using (1.2), we obtain

(A.3)
$$\| \nabla v_0^\varepsilon \|_{L^2(I_\delta)} \le C \| \nabla V_l \|_{L^2(I_\delta)}.$$

A direct calculation gives

$$\mathrm{div}(A^\varepsilon \nabla v_0^\varepsilon) = \mathrm{div}((A^\varepsilon)^\top) \cdot \nabla v_0^\varepsilon + A^\varepsilon : \nabla^2 v_0^\varepsilon = \mathrm{div}((A^\varepsilon)^\top) \cdot \nabla v_0^\varepsilon + A_S^\varepsilon : \nabla^2 v_0^\varepsilon,$$

where the symmetric part is defined as $A_S^\varepsilon := \dfrac{1}{2}\Big(A^\varepsilon + (A^\varepsilon)^\top\Big)$. Therefore,

$$A_S^\varepsilon(\boldsymbol{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} = A^\varepsilon(\boldsymbol{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \ge \lambda |\boldsymbol{\xi}|^2.$$

We rewrite (A.1) as

$$\begin{cases} -A_S^\varepsilon : \nabla^2 (v_0^\varepsilon - V_l) = \mathrm{div}((A^\varepsilon)^\top) \cdot \nabla v_0^\varepsilon & \text{in } I_\delta, \\ v_0^\varepsilon - V_l = 0 & \text{on } \partial I_\delta. \end{cases}$$

According to [12, Appendix A] for the symmetric matrix A_S^ε,

$$\| \nabla^2 v_0^\varepsilon \|_{L^2(I_\delta)} \le C\Big(\| \nabla A_S^\varepsilon \|_{L^\infty(I_\delta)} \| \nabla(v_0^\varepsilon - V_l) \|_{L^2(I_\delta)} + \| \mathrm{div}((A^\varepsilon)^\top) \cdot \nabla v_0^\varepsilon \|_{L^2(I_\delta)} \Big)$$

$$\le C\varepsilon^{-1} \| \nabla V_l \|_{L^2(I_\delta)},$$

where we have used (A.3) in the last step. This gives (A.2). $\qquad\square$

687   Next we estimate $v^\varepsilon - v_0^\varepsilon$.

688   LEMMA A.2. *Let $\iota = \mu\varepsilon^\gamma$ with $\gamma > 1$ and $\|\nabla_{\boldsymbol{y}} A\|_{L^\infty(\Omega \times Y)}$ is bounded, and v^ε is*
689   *the solution of* (2.3) *with the essential boundary conditions for $d = 2, 3$, then*

690   (A.4) $\|\nabla(v^\varepsilon - v_0^\varepsilon)\|_{L^2(I_\delta)} \le C\varepsilon^{(\gamma-1)/2}\|\nabla V_l\|_{L^2(I_\delta)}$,

691   *and*

692   (A.5) $\|\nabla v^\varepsilon\|_\iota \le C\varepsilon^{-(1+\gamma)/2}\|\nabla V_l\|_{L^2(I_\delta)}$.

693   *Proof.* Using (A.1), we rewrite (2.3) with the essential boundary conditions as

694
$$\begin{cases} \Delta^2(v^\varepsilon - V_l) = \iota^{-2}\mathrm{div}(A^\varepsilon \nabla(v^\varepsilon - v_0^\varepsilon)) & \text{in } I_\delta, \\ v^\varepsilon - V_l = \partial_{\boldsymbol{n}}(v^\varepsilon - V_l) = 0 & \text{on } \partial I_\delta. \end{cases}$$

695   Scaling by $\hat{\varepsilon}: = \varepsilon/\delta, \hat{\iota}: = \iota/\delta, \hat{\boldsymbol{x}}: = \boldsymbol{x}/\delta$ and with the notation $\hat{v}^\varepsilon(\hat{\boldsymbol{x}}): = v^\varepsilon(\boldsymbol{x}), \hat{V}_l(\hat{\boldsymbol{x}}): =$
696      $V_l(\boldsymbol{x}), \hat{v}_0^\varepsilon(\hat{\boldsymbol{x}}): = v_0^\varepsilon(\boldsymbol{x})$, we rewrite the above equation as the boundary value problems

697
$$\begin{cases} \Delta^2(\hat{v}^\varepsilon - \hat{V}_l) = \hat{\iota}^{-2}\mathrm{div}(A^\varepsilon(\delta\cdot)\nabla(\hat{v}^\varepsilon - \hat{v}_0^\varepsilon)) & \text{in } I_1, \\ \hat{v}^\varepsilon - \hat{V}_l = \partial_{\boldsymbol{n}}(\hat{v}^\varepsilon - \hat{V}_l) = 0 & \text{on } \partial I_1. \end{cases}$$

698   By [24, Theorem 4.3.10], we get

699      $\|\hat{v}^\varepsilon - \hat{V}_l\|_{H^3(I_1)} \le C\hat{\iota}^{-2}\|\mathrm{div}(A^\varepsilon(\delta\cdot)\nabla(\hat{v}^\varepsilon - \hat{v}_0^\varepsilon))\|_{H^{-1}(I_1)} \le C\hat{\iota}^{-2}\|\nabla(\hat{v}^\varepsilon - \hat{v}_0^\varepsilon)\|_{L^2(I_1)}$.

700   Rescaling back to T_δ, we obtain

701   (A.6)
$$\begin{aligned} \|\nabla^3 v^\varepsilon\|_{L^2(I_\delta)} &\le C\delta^{d/2-3}\|\nabla^3 \hat{v}^\varepsilon\|_{L^2(I_1)} \le C\delta^{d/2-1}\iota^{-2}\|\nabla(\hat{v}^\varepsilon - \hat{v}_0^\varepsilon)\|_{L^2(I_1)} \\ &\le C\iota^{-2}\|\nabla(v^\varepsilon - v_0^\varepsilon)\|_{L^2(I_\delta)}. \end{aligned}$$

702   Proceeding along the same line of [27, Theorem 5.2], using the definitions of (2.3)
703   and (A.1), for any $z \in H_0^1(I_\delta) \cap H^2(I_\delta)$, we write

704
$$\iota^2(\nabla^2 v^\varepsilon, \nabla^2 z)_{I_\delta} + (A^\varepsilon\nabla(v^\varepsilon - v_0^\varepsilon), \nabla z)_{I_\delta} = \iota^2\int_{\partial I_\delta} \partial_{\boldsymbol{n}}^2 v^\varepsilon \partial_{\boldsymbol{n}} z\mathrm{d}\sigma(\boldsymbol{x}).$$

705   Choosing $z = v^\varepsilon - v_0^\varepsilon$ in the above identity, we obtain

706      $\iota^2\|\nabla^2 v^\varepsilon\|_{L^2(I_\delta)}^2 + \|\nabla(v^\varepsilon - v_0^\varepsilon)\|_{L^2(I_\delta)}^2 \le \iota^2(\nabla^2 v^\varepsilon, \nabla^2 v_0^\varepsilon)_{I_\delta} - \iota^2\int_{\partial I_\delta} \partial_{\boldsymbol{n}}^2 v^\varepsilon \partial_{\boldsymbol{n}} v_0^\varepsilon\mathrm{d}\sigma(\boldsymbol{x}).$

707   The first term may be bounded by (A.2), it remains to estimate the boundary
708   term. According to the trace inequality and (A.6),

709      $\iota^{3/2}\|\partial_{\boldsymbol{n}}^2 v^\varepsilon\|_{L^2(\partial I_\delta)} \le C\iota^{3/2}\left(\delta^{-1/2}\|\partial_{\boldsymbol{n}}^2 v^\varepsilon\|_{L^2(I_\delta)} + \|\partial_{\boldsymbol{n}}^2 v^\varepsilon\|_{L^2(I_\delta)}^{1/2}\|\nabla^3 v^\varepsilon\|_{L^2(I_\delta)}^{1/2}\right)$

710      $\le C(\iota\|\nabla^2 v^\varepsilon\|_{L^2(I_\delta)} + \|\nabla(v^\varepsilon - v_0^\varepsilon)\|_{L^2(I_\delta)})$,
711

712   where we have used the relation $\iota \ll \varepsilon \ll \delta$.
713   Invoking the trace inequality again and using (A.2), we obtain

714      $\iota^{1/2}\|\partial_{\boldsymbol{n}} v_0^\varepsilon\|_{L^2(\partial I_\delta)} \le C\iota^{1/2}\left(\delta^{-1/2}\|\partial_{\boldsymbol{n}} v_0^\varepsilon\|_{L^2(I_\delta)} + \|\partial_{\boldsymbol{n}} v_0^\varepsilon\|_{L^2(I_\delta)}^{1/2}\|\nabla^2 v_0^\varepsilon\|_{L^2(I_\delta)}^{1/2}\right)$

715      $\le C\sqrt{\iota/\varepsilon}\|\nabla V_l\|_{L^2(I_\delta)}$

716      $\le C\varepsilon^{(\gamma-1)/2}\|\nabla V_l\|_{L^2(I_\delta)}$,
717

718   where we have used the fact $\varepsilon \ll \delta$. Summing up all the above estimates, we get

719
$$\iota \| \nabla^2 v^\varepsilon \|_{L^2(I_\delta)} + \| \nabla(v^\varepsilon - v_0^\varepsilon) \|_{L^2(I_\delta)} \le C\varepsilon^{(\gamma-1)/2} \| \nabla V_l \|_{L^2(I_\delta)},$$

720   This gives (A.4) and the H^2-estimate of (A.5).

721   Substituting (A.4) into (A.6) we obtain the H^3−estimate of (A.5).   □

722   Interpolate (A.4) and (A.5) with (A.2), we get

723
$$\| v^\varepsilon \|_{H^{3/2}(I_\delta)} + \iota \| v^\varepsilon \|_{H^{5/2}(I_\delta)} \le C\varepsilon^{-1/2} \| \nabla V_l \|_{L^2(I_\delta)}.$$

724   By [21, Theorem 3], there exists a regularized interpolation operator $I_h : H^1(I_\delta) \to X_h$
725 such that for any $v \in H^s(I_\delta), 1 \le s \le 3$,

726
$$\| \nabla^j(I - I_h)v \|_{L^2(I_\delta)} \le Ch^{s-j} \| \nabla^s v \|_{L^2(I_\delta)}, \qquad 0 \le j \le s,$$

727   where j is a non-negative integer, which implies

728
$$\| v^\varepsilon - I_h v^\varepsilon \|_{\iota,h} \le Ch^{1/2}(\| v^\varepsilon \|_{H^{3/2}(I_\delta)} + \iota \| v^\varepsilon \|_{H^{5/2}(I_\delta)}) \le C\sqrt{h/\varepsilon} \| \nabla V_l \|_{L^2(I_\delta)}.$$

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