

A New Exponential Compact Scheme for the Two-Dimensional Unsteady Nonlinear Burgers' and Navier-Stokes Equations in Polar Cylindrical Coordinates

R.K. Mohanty^{1,*}, Li Yuan^{2,3} and Divya Sharma¹

¹ Department of Mathematics, South Asian University, Akbar Bhawan, Chanakyapuri, New Delhi-110021, India

² LSEC and ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, No. 55-Zhong Guan Cun East Road, Beijing 100190, China

³ School of Mathematical Sciences, University of Chinese Academy of Sciences, No. 55-Zhong Guan Cun East Road, Beijing 100190, China

Received 1 April 2020; Accepted (in revised version) 1 October 2020

Abstract. In this article, a new compact difference scheme is proposed in exponential form to solve two-dimensional unsteady nonlinear Burgers' and Navier-Stokes equations of motion in polar cylindrical coordinates by using half-step discretization. At each time level by using only nine grid points in space, the proposed scheme gives accuracy of order four in space and two in time. The method is directly applicable to the equations having singularities at boundary points. Stability analysis is explained in detail and many benchmark problems like Burgers', Navier-Stokes and Taylor-vortex problems in polar cylindrical coordinates are solved to verify the accuracy and efficiency of the scheme.

AMS subject classifications: 65M06, 65M10, 65Z05, 65Y99

Key words: Half-step discretization, compact scheme in exponential form, two-level implicit scheme, Burgers' equation, Navier-Stokes equations of motion, Taylor-vortex problem.

1. Introduction

Nonlinear parabolic partial differential equations (PDEs) arise in various branches of physical and engineering sciences and play a significant role in computational fluid dynamics (CFD). Two of the most important examples of nonlinear PDEs are the Burg-

*Corresponding author. *Email addresses:* rmohanty@sau.ac.in (R.K. Mohanty), lyuan@lsec.cc.ac.cn (L. Yuan), udivya93@gmail.com (D. Sharma)

ers' and Navier-Stokes (N-S) equations [3] which have a large variety of applications in studying viscous flow and turbulence and in modeling weather [24]. The Burgers' equation is a special type of nonlinear partial differential equation which is very similar to the N-S equations and could serve as a prototype of the N-S equations. N-S equations are very famous and primordial governing equations in fluid mechanics. Many physical incompressible fluid flow problems are modeled by viscous incompressible N-S equations and numerical computation of these equations is a crucial tool.

During past few decades, tremendous efforts have been made for the development of computational schemes of different numerical approaches to find better approximations of these nonlinear partial differential equations. Finite Difference Method (FDM) [10,14,25] is a very popular numerical approach and many important numerical techniques based on FDM have been developed to solve 2D unsteady nonlinear Burgers' [7] and N-S equations in the polar cylindrical coordinate system. In [2,4,9,12,16] several high order finite difference schemes are proposed to solve 2D unsteady linear parabolic equations numerically depending on alternating direction implicit (ADI) technique. Later, Mohanty and Setia [20] solved the system of quasi-linear parabolic PDEs by using half-step discretization. In order to solve the Navier-Stokes equations for very small Reynolds number, Eren [5] applied the Crank-Nicolson implicit method for the solution of 2D time dependent flow field for interaction of jets. Mohanty *et al.* [15] have developed an operator compact implicit scheme for 1D viscous Burgers-Huxley equation by using three spatial uniform grids and two levels of time. The method has order of accuracy two in time and four in space. Bai *et al.* [1] presented and analyzed approximated ILU and UGS preconditioning methods for the linearized discretized steady incompressible N-S equations. Shah *et al.* [23] proposed a third order upwind compact scheme for the incompressible N-S equations in the artificial compressibility method in general curvilinear coordinates. Finite volume method (FVM) is one of the well-known attractive approaches to tackle with PDEs in complex domain. Pereira *et al.* [22] constructed a fourth order accurate compact scheme based on finite volume method for the incompressible N-S equations in primitive variable formulation. In the field of computational fluid dynamics, high order compact (HOC) formulations are becoming more popular because they provide more accurate solutions in a compact stencil. A class of implicit HOC finite difference schemes have been developed by Kalita *et al.* [11] with weighted time discretization to solve the unsteady 2D variable coefficient convection-diffusion equation. Their methods were tested on problems with both Dirichlet and Neumann boundary conditions. Erturk and Gokcol [6] introduced a new compact formulation of order four in order to solve steady 2D incompressible Navier-Stokes equations. Recently, Mohanty *et al.* [17–19,21] proposed compact HOC method in exponential form in order to solve nonlinear boundary value problems (BVPs).

To our best knowledge no method in exponential form having order accuracy two in time and four in space for the solution of 2D nonlinear parabolic PDEs in cylindrical polar coordinates has been discussed in the literature so far. In this article, we discuss a new half-step discretization in exponential form for the solution of 2D unsteady nonlinear Burgers' and N-S equations in polar cylindrical coordinates. The proposed

method is second order accurate in time and fourth order accurate in space, and on each time level only nine grid points are used for a single compact cell. The prominent advantage of developing the method is that the problems having singularities in the domain can be solved directly.

The paper is arranged as follows. In Section 2 we present the new off step discretization schemes and its derivation for the solution of the nonlinear 2D parabolic equations in r - z plane. In Section 3 we discuss the application to the Burgers' and N-S equations. In Section 4, brief discussion of unconditionally stable character of the proposed method is given and several numerical experiments to test the accuracy and efficiency of the proposed method are given in Section 5. Section 6 contains concluding remarks.

2. Discretization procedure

We consider the 2D nonlinear parabolic equation in the r - z plane.

$$\nu \left(\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} \right) = \frac{\partial \omega}{\partial t} + \phi \left(r, z, t, \omega, \frac{\partial \omega}{\partial r}, \frac{\partial \omega}{\partial z} \right), \quad 0 < r, z < 1, \quad t > 0 \tag{2.1}$$

defined in the semi-infinite region $\Omega = \{(r, z, t) \mid 0 < r, z < 1, t > 0\}$, where $\nu > 0$.

The initial and Dirichlet boundary conditions are given by

$$\omega(r, z, 0) = f(r, z), \quad 0 \leq r, z \leq 1, \tag{2.2}$$

$$\omega(r, 0, t) = a_0(r, t), \quad \omega(r, 1, t) = a_1(r, t), \quad 0 \leq r \leq 1, \quad t > 0, \tag{2.3}$$

$$\omega(0, z, t) = b_0(z, t), \quad \omega(1, z, t) = b_1(z, t), \quad 0 \leq z \leq 1, \quad t > 0, \tag{2.4}$$

where $\omega(r, z, t) \in C^6(\Omega)$, $\phi \geq 0$, and $f(r, z), a_0(r, t), a_1(r, t), b_0(z, t)$ and $b_1(z, t)$ are smooth functions on the boundary of Ω . Further, we assume that the solution is unique and the solution domain under consideration is regular, that is, there is no discontinuities in the solution domain.

We discretized the domain Ω into rectangular grids. $h > 0$ is mesh size in r - and z -directions, $\tau > 0$ is mesh size in t -direction. Each internal grid point is denoted by (r_l, z_m, t_j) , where $r_l = lh, z_m = mh$, and $t_j = j\tau$, with $l, m = 0, \dots, N + 1$ and $j = 0, 1, \dots$, where N be a positive integer defined by $(N + 1)h = 1$. Right half step point of r_l is defined as $r_{l+\frac{1}{2}} = r_l + \frac{h}{2}$ and left half step point of r_l is defined as $r_{l-\frac{1}{2}} = r_l - \frac{h}{2}$. The mesh ratio parameter is given by $\lambda = \frac{\tau}{h^2}$. Let $W_{l,m}^j$ be the exact solution value of $\omega(r, z, t)$ at the grid point (r_l, z_m, t_j) and is approximated by $\omega_{l,m}^j$.

At the grid point (r_l, z_m, t_j) the differential equation (2.1) may be written as

$$\nu \left(\frac{\partial^2 \omega_{l,m}^j}{\partial r^2} + \frac{\partial^2 \omega_{l,m}^j}{\partial z^2} \right) - \frac{\partial \omega_{l,m}^j}{\partial t} = \phi \left(r_l, z_m, t_j, \omega_{l,m}^j, \frac{\partial \omega_{l,m}^j}{\partial r}, \frac{\partial \omega_{l,m}^j}{\partial z} \right) \equiv \phi_{l,m}^j. \tag{2.5}$$

Let

$$\bar{t}_j = t_j + \frac{\tau}{2}, \tag{2.6}$$

$$\bar{\omega}_{l,m}^j = \frac{1}{2} \left(\omega_{l,m}^{j+1} + \omega_{l,m}^j \right), \quad (2.7)$$

$$\bar{\omega}_{l\pm 1,m}^j = \frac{1}{2} \left(\omega_{l\pm 1,m}^{j+1} + \omega_{l\pm 1,m}^j \right), \quad (2.8)$$

$$\bar{\omega}_{l,m\pm 1}^j = \frac{1}{2} \left(\omega_{l,m\pm 1}^{j+1} + \omega_{l,m\pm 1}^j \right), \quad (2.9)$$

$$\bar{\omega}_{l\pm \frac{1}{2},m}^j = \frac{1}{2} \left(\bar{\omega}_{l\pm 1,m}^j + \bar{\omega}_{l,m}^j \right), \quad (2.10)$$

$$\bar{\omega}_{l,m\pm \frac{1}{2}}^j = \frac{1}{2} \left(\bar{\omega}_{l,m\pm 1}^j + \bar{\omega}_{l,m}^j \right), \quad (2.11)$$

$$\bar{\omega}_{rl,m}^j = \frac{1}{2h} \left(\bar{\omega}_{l+1,m}^j - \bar{\omega}_{l-1,m}^j \right), \quad (2.12)$$

$$\bar{\omega}_{rl+\frac{1}{2},m}^j = \frac{1}{h} \left(\bar{\omega}_{l+1,m}^j - \bar{\omega}_{l,m}^j \right), \quad (2.13)$$

$$\bar{\omega}_{rl-\frac{1}{2},m}^j = \frac{1}{h} \left(\bar{\omega}_{l,m}^j - \bar{\omega}_{l-1,m}^j \right), \quad (2.14)$$

$$\bar{\omega}_{rl,m+\frac{1}{2}}^j = \frac{1}{4h} \left(\bar{\omega}_{l+1,m+1}^j - \bar{\omega}_{l-1,m+1}^j + \bar{\omega}_{l+1,m}^j - \bar{\omega}_{l-1,m}^j \right), \quad (2.15)$$

$$\bar{\omega}_{rl,m-\frac{1}{2}}^j = \frac{1}{4h} \left(\bar{\omega}_{l+1,m-1}^j - \bar{\omega}_{l-1,m-1}^j + \bar{\omega}_{l+1,m}^j - \bar{\omega}_{l-1,m}^j \right), \quad (2.16)$$

$$\bar{\omega}_{zl,m}^j = \frac{1}{2h} \left(\bar{\omega}_{l,m+1}^j - \bar{\omega}_{l,m-1}^j \right), \quad (2.17)$$

$$\bar{\omega}_{zl,m+\frac{1}{2}}^j = \frac{1}{h} \left(\bar{\omega}_{l,m+1}^j - \bar{\omega}_{l,m}^j \right), \quad (2.18)$$

$$\bar{\omega}_{zl,m-\frac{1}{2}}^j = \frac{1}{h} \left(\bar{\omega}_{l,m}^j - \bar{\omega}_{l,m-1}^j \right), \quad (2.19)$$

$$\bar{\omega}_{zl+\frac{1}{2},m}^j = \frac{1}{4h} \left(\bar{\omega}_{l+1,m+1}^j - \bar{\omega}_{l+1,m-1}^j + \bar{\omega}_{l,m+1}^j - \bar{\omega}_{l,m-1}^j \right), \quad (2.20)$$

$$\bar{\omega}_{zl-\frac{1}{2},m}^j = \frac{1}{4h} \left(\bar{\omega}_{l-1,m+1}^j - \bar{\omega}_{l-1,m-1}^j + \bar{\omega}_{l,m+1}^j - \bar{\omega}_{l,m-1}^j \right), \quad (2.21)$$

$$\bar{\omega}_{t,l,m}^j = \frac{1}{\tau} \left(\omega_{l,m}^{j+1} - \omega_{l,m}^j \right), \quad (2.22)$$

$$\bar{\omega}_{t,l\pm 1,m}^j = \frac{1}{\tau} \left(\omega_{l\pm 1,m}^{j+1} - \omega_{l\pm 1,m}^j \right), \quad (2.23)$$

$$\bar{\omega}_{t,l,m\pm 1}^j = \frac{1}{\tau} \left(\omega_{l,m\pm 1}^{j+1} - \omega_{l,m\pm 1}^j \right), \quad (2.24)$$

$$\bar{\omega}_{t,l\pm \frac{1}{2},m}^j = \frac{1}{2\tau} \left(\omega_{l\pm 1,m}^{j+1} + \omega_{l,m}^{j+1} - \omega_{l\pm 1,m}^j - \omega_{l,m}^j \right), \quad (2.25)$$

$$\bar{\omega}_{t,l,m\pm \frac{1}{2}}^j = \frac{1}{2\tau} \left(\omega_{l,m\pm 1}^{j+1} + \omega_{l,m}^{j+1} - \omega_{l,m\pm 1}^j - \omega_{l,m}^j \right), \quad (2.26)$$

$$\bar{\omega}_{rrl,m}^j = \frac{1}{h^2} \left(\bar{\omega}_{l+1,m}^j - 2\bar{\omega}_{l,m}^j + \bar{\omega}_{l-1,m}^j \right), \quad (2.27)$$

$$\bar{\omega}_{zzl,m}^j = \frac{1}{h^2} \left(\bar{\omega}_{l,m+1}^j - 2\bar{\omega}_{l,m}^j + \bar{\omega}_{l,m-1}^j \right), \quad (2.28)$$

$$\bar{\omega}_{rrl,m\pm 1}^j = \frac{1}{h^2} \left(\bar{\omega}_{l+1,m\pm 1}^j - 2\bar{\omega}_{l,m\pm 1}^j + \bar{\omega}_{l-1,m\pm 1}^j \right), \tag{2.29}$$

$$\bar{\omega}_{zzl,\pm 1,m}^j = \frac{1}{h^2} \left(\bar{\omega}_{l\pm 1,m+1}^j - 2\bar{\omega}_{l\pm 1,m}^j + \bar{\omega}_{l\pm 1,m-1}^j \right), \tag{2.30}$$

$$\bar{\omega}_{rrrtl,m}^j = \frac{1}{h^2} \left(\bar{\omega}_{tl+1,m}^j - 2\bar{\omega}_{tl,m}^j + \bar{\omega}_{tl-1,m}^j \right), \tag{2.31}$$

$$\bar{\omega}_{zzztl,m}^j = \frac{1}{h^2} \left(\bar{\omega}_{tl,m+1}^j - 2\bar{\omega}_{tl,m}^j + \bar{\omega}_{tl,m-1}^j \right). \tag{2.32}$$

The central and averaging operators are defined by

$$\begin{aligned} \delta_r w_{l,m}^j &= \left(w_{l+\frac{1}{2},m}^j - w_{l-\frac{1}{2},m}^j \right), \\ \mu_r w_{l,m}^j &= \frac{1}{2} \left(w_{l+\frac{1}{2},m}^j + w_{l-\frac{1}{2},m}^j \right), \dots, \text{ etc.} \end{aligned}$$

Expanding exponential function and Taylor series, we can write

$$\begin{aligned} &\nu \left(\delta_r^2 + \delta_z^2 + \frac{1}{6} \delta_r^2 \delta_z^2 \right) \omega_{l,m}^j \tag{2.33} \\ &= h^2 \omega_{l,m}^j \exp \left\{ \frac{1}{3\omega_{l,m}^j} \left(\omega_{l+\frac{1}{2},m}^j + \omega_{l-\frac{1}{2},m}^j + \omega_{l,m+\frac{1}{2}}^j + \omega_{l,m-\frac{1}{2}}^j - 4\omega_{l,m}^j \right) \right\} \\ &+ h^2 \phi_{l,m}^j \exp \left[\frac{1}{3\phi_{l,m}^j} \left(\phi_{l+\frac{1}{2},m}^j + \phi_{l-\frac{1}{2},m}^j + \phi_{l,m+\frac{1}{2}}^j + \phi_{l,m-\frac{1}{2}}^j - 4\phi_{l,m}^j \right) \right] + \mathcal{O}(h^6). \end{aligned}$$

At the grid point (r_l, z_m, t_j) , we denote

$$\omega_{abc} = \frac{\partial^{a+b+c} \omega_{l,m}^j}{\partial r^a \partial z^b \partial t^c}, \quad a, b, c = 0, 1, \dots, \tag{2.34}$$

$$\alpha = (\phi_t)_{l,m}^j, \quad \beta = (\phi_\omega)_{l,m}^j, \quad \gamma = (\phi_{\omega_r})_{l,m}^j, \quad \eta = \phi_{\omega_z}^j. \tag{2.35}$$

Differentiating Eq. (2.1) with respect to t and by the help of (2.34) and (2.35), we obtain a relation

$$\nu(\omega_{201} + \omega_{021}) = \omega_{002} + \alpha + \beta\omega_{001} + \gamma\omega_{101} + \eta\omega_{011}. \tag{2.36}$$

Next, we define the approximations

$$\bar{\phi}_{l\pm\frac{1}{2},m}^j = \phi \left(r_{l\pm\frac{1}{2}}, z_m, \bar{t}_j, \bar{\omega}_{l\pm\frac{1}{2},m}^j, \bar{\omega}_{r_{l\pm\frac{1}{2},m}}^j, \bar{\omega}_{z_{l\pm\frac{1}{2},m}}^j \right), \tag{2.37}$$

$$\bar{\phi}_{l,m\pm\frac{1}{2}}^j = \phi \left(r_l, z_{m\pm\frac{1}{2}}, \bar{t}_j, \omega_{l,m\pm\frac{1}{2}}^j, \bar{\omega}_{r_{l,m\pm\frac{1}{2}}}^j, \bar{\omega}_{z_{l,m\pm\frac{1}{2}}}^j \right). \tag{2.38}$$

By the help of the approximations (2.6)-(2.21) and the notations (2.34) and (2.35), simplifying (2.37) and (2.38), we obtain

$$\bar{\phi}_{l\pm\frac{1}{2},m}^j = \phi_{l\pm\frac{1}{2},m}^j + \frac{\tau}{2} (\alpha + \beta\omega_{001} + \gamma\omega_{101} + \eta\omega_{011}) + \frac{h^2}{24} T_1 \pm \mathcal{O}(\tau h + h^3), \tag{2.39}$$

$$\bar{\phi}_{l,m\pm\frac{1}{2}}^j = \phi_{l,m\pm\frac{1}{2}}^j + \frac{\tau}{2} (\alpha + \beta\omega_{001} + \gamma\omega_{101} + \eta\omega_{011}) + \frac{h^2}{24} T_2 \pm \mathcal{O}(\tau h + h^3), \tag{2.40}$$

where

$$\begin{aligned} T_1 &= 3\beta\omega_{200} + \gamma\omega_{300} + 3\eta\omega_{210} + 4\eta\omega_{030}, \\ T_2 &= 3\beta\omega_{020} + 3\gamma\omega_{120} + 4\gamma\omega_{300} + \eta\omega_{030}. \end{aligned}$$

Next we define

$$\begin{aligned} \bar{\bar{\omega}}_{rl,m}^j &= \bar{\omega}_{rl,m}^j + a_1 h \left[\left(\bar{\omega}_{t_{l+\frac{1}{2},m}}^j - \bar{\omega}_{t_{l-\frac{1}{2},m}}^j \right) + \left(\bar{\phi}_{l+\frac{1}{2},m}^j - \bar{\phi}_{l-\frac{1}{2},m}^j \right) \right] \\ &\quad + a_2 h \left(\bar{\omega}_{zz_{l+1,m}}^j - \bar{\omega}_{zz_{l-1,m}}^j \right), \end{aligned} \tag{2.41}$$

$$\begin{aligned} \bar{\bar{\omega}}_{zl,m}^j &= \bar{\omega}_{zl,m}^j + b_1 h \left[\left(\bar{\omega}_{t_{l,m+\frac{1}{2}}}^j - \bar{\omega}_{t_{l,m-\frac{1}{2}}}^j \right) + \left(\bar{\phi}_{l,m+\frac{1}{2}}^j - \bar{\phi}_{l,m-\frac{1}{2}}^j \right) \right] \\ &\quad + b_2 h \left(\bar{\omega}_{rr_{l,m+1}}^j - \bar{\omega}_{rr_{l,m-1}}^j \right), \end{aligned} \tag{2.42}$$

where a_1, a_2, b_1 and b_2 are parameters to be determined.

By the help of the approximations (2.6)-(2.26), (2.39) and (2.40), the approximations (2.41) and (2.2) simplify to

$$\begin{aligned} \bar{\bar{\omega}}_{rl,m}^j &= \omega_{rl,m}^j + \frac{\tau}{2}\omega_{101} + \frac{h^2}{6}(1 + 6va_1)\omega_{300} \\ &\quad + h^2(va_1 + 2a_2)\omega_{120} + \mathcal{O}(\tau^2 + \tau h^2 + h^4), \end{aligned} \tag{2.43}$$

$$\begin{aligned} \bar{\bar{\omega}}_{zl,m}^j &= \omega_{zl,m}^j + \frac{\tau}{2}\omega_{011} + \frac{h^2}{6}(1 + 6vb_1)\omega_{030} \\ &\quad + h^2(vb_1 + 2b_2)\omega_{210} + \mathcal{O}(\tau^2 + \tau h^2 + h^4). \end{aligned} \tag{2.44}$$

For the values $a_1 = b_1 = -\frac{1}{6\nu}, a_2 = b_2 = \frac{1}{12}$ from (2.43) and (2.44), it is easy to verify that

$$\bar{\bar{\omega}}_{rl,m}^j = \omega_{rl,m}^j + \frac{\tau}{2}\omega_{101} + \mathcal{O}(\tau^2 + \tau h^2 + h^4), \tag{2.45}$$

$$\bar{\bar{\omega}}_{zl,m}^j = \omega_{zl,m}^j + \frac{\tau}{2}\omega_{011} + \mathcal{O}(\tau^2 + \tau h^2 + h^4). \tag{2.46}$$

Next we define,

$$\bar{\bar{\phi}}_{l,m}^j = \phi \left(r_l, z_m, \bar{t}_j, \bar{\omega}_{l,m}^j, \bar{\bar{\omega}}_{rl,m}^j, \bar{\bar{\omega}}_{zl,m}^j \right). \tag{2.47}$$

Using the previous approximations (2.6), (2.7), (2.45), (2.46), Eq. (2.47) reduces to

$$\bar{\bar{\phi}}_{l,m}^j = \phi_{l,m}^j + \frac{\tau}{2}(\alpha + \beta\omega_{001} + \gamma\omega_{101} + \eta\omega_{011}) + \mathcal{O}(\tau^2 + \tau h^2 + h^4). \tag{2.48}$$

Let

$$\begin{aligned} \hat{\bar{\omega}}_{rl,m}^j &= \bar{\omega}_{rl,m}^j + a_3 h \left[\left(\bar{\omega}_{t_{l+\frac{1}{2},m}}^j - \bar{\omega}_{t_{l-\frac{1}{2},m}}^j \right) + \left(\bar{\phi}_{l+\frac{1}{2},m}^j - \bar{\phi}_{l-\frac{1}{2},m}^j \right) \right] \\ &\quad + a_4 h \left(\bar{\omega}_{zz_{l+1,m}}^j - \bar{\omega}_{zz_{l-1,m}}^j \right), \end{aligned} \tag{2.49}$$

$$\begin{aligned} \widehat{\omega}_{z,l,m}^j &= \overline{\omega}_{z,l,m}^j + b_3 h \left[\left(\overline{\omega}_{t,l,m+\frac{1}{2}}^j - \overline{\omega}_{t,l,m-\frac{1}{2}}^j \right) + \left(\overline{\phi}_{l,m+\frac{1}{2}}^j - \overline{\phi}_{l,m-\frac{1}{2}}^j \right) \right] \\ &\quad + b_4 h \left(\overline{\omega}_{rr,l,m+1}^j - \overline{\omega}_{rr,l,m-1}^j \right), \end{aligned} \tag{2.50}$$

$$\widehat{\omega}_{l,m}^j = \overline{\omega}_{l,m}^j + c_3 h^2 \overline{\omega}_{rr,l,m}^j + c_4 h^2 \overline{\omega}_{zz,l,m}^j, \tag{2.51}$$

$$\widehat{\omega}_{t,l,m}^j = \overline{\omega}_{t,l,m}^j + d_3 h^2 \overline{\omega}_{rr,t,l,m}^j + d_4 h^2 \overline{\omega}_{zz,t,l,m}^j, \tag{2.52}$$

where $a_3, a_4, b_3, b_4, c_3, c_4, d_3$ and d_4 are parameters to be estimated.

On simplifying (2.49)-(2.52), we get

$$\widehat{\omega}_{r,l,m}^j = \omega_{r,l,m}^j + \frac{\tau}{2} \omega_{101} + \frac{h^2}{6} T_3 + \mathcal{O}(\tau^2 + \tau h^2 + h^4), \tag{2.53}$$

$$\widehat{\omega}_{z,l,m}^j = \omega_{z,l,m}^j + \frac{\tau}{2} \omega_{011} + \frac{h^2}{6} T_4 + \mathcal{O}(\tau^2 + \tau h^2 + h^4), \tag{2.54}$$

$$\widehat{\omega}_{l,m}^j = \omega_{l,m}^j + \frac{\tau}{2} \omega_{001} + \frac{h^2}{6} T_5 + \mathcal{O}(\tau^2 + \tau h^2 + h^4), \tag{2.55}$$

$$\widehat{\omega}_{t,l,m}^j = \omega_{t,l,m}^j + \frac{\tau}{2} \omega_{002} + \frac{h^2}{6} T_6 + \mathcal{O}(\tau^2 + \tau h^2 + h^4), \tag{2.56}$$

where

$$T_3 = (1 + 6\nu a_3) \omega_{300} + 6(\nu a_3 + 2a_4) \omega_{120},$$

$$T_4 = (1 + 6\nu b_3) \omega_{030} + 6(\nu b_3 + 2b_4) \omega_{210},$$

$$T_5 = 6c_3 \omega_{200} + 6c_4 \omega_{020},$$

$$T_6 = 6d_3 \omega_{201} + 6d_4 \omega_{021}.$$

Now, we define a new approximation

$$\widehat{\phi}_{l,m}^j = \phi \left(r_l, z_m, \bar{t}_j, \widehat{\omega}_{l,m}^j, \widehat{\omega}_{r,l,m}^j, \widehat{\omega}_{z,l,m}^j \right). \tag{2.57}$$

Simplifying (2.57) by the help of (2.6), (2.53)-(2.55), we get

$$\begin{aligned} \widehat{\phi}_{l,m}^j &= \phi_{l,m}^j + \frac{\tau}{2} (\alpha + \beta \omega_{001} + \gamma \omega_{101} + \eta \omega_{011}) \\ &\quad + \frac{h^2}{6} (\beta T_5 + \gamma T_3 + \eta T_4) + \mathcal{O}(\tau^2 + \tau h^2 + h^4). \end{aligned} \tag{2.58}$$

Note that,

$$\delta_r^2 \overline{\omega}_{l,m}^j = \delta_r^2 \omega_{l,m}^j + \frac{\tau h^2}{2} \omega_{201} + \mathcal{O}(\tau^2 h^2 + \tau h^4), \tag{2.59}$$

$$\delta_z^2 \overline{\omega}_{l,m}^j = \delta_z^2 \omega_{l,m}^j + \frac{\tau h^2}{2} \omega_{021} + \mathcal{O}(\tau^2 h^2 + \tau h^4), \tag{2.60}$$

$$\delta_r^2 \delta_z^2 \overline{\omega}_{l,m}^j = \delta_r^2 \delta_z^2 \omega_{l,m}^j + \mathcal{O}(\tau h^4). \tag{2.61}$$

Then at each internal node (r_l, z_m, t_j) , the given differential equation (2.1) is discretized by

$$\begin{aligned} & \nu \left(\delta_r^2 + \delta_z^2 + \frac{1}{6} \delta_r^2 \delta_z^2 \right) \bar{\omega}_{l,m}^j \tag{2.62} \\ &= h^2 \bar{\omega}_{l,m}^j \exp \left[\frac{1}{3 \bar{\omega}_{l,m}^j} \left(\bar{\omega}_{l+\frac{1}{2},m}^j + \bar{\omega}_{l-\frac{1}{2},m}^j + \bar{\omega}_{l,m+\frac{1}{2}}^j + \bar{\omega}_{l,m-\frac{1}{2}}^j - 4 \widehat{\bar{\omega}}_{l,m}^j \right) \right] \\ &+ h^2 \bar{\phi}_{l,m}^j \exp \left[\frac{1}{3 \bar{\phi}_{l,m}^j} \left(\bar{\phi}_{l+\frac{1}{2},m}^j + \bar{\phi}_{l-\frac{1}{2},m}^j + \bar{\phi}_{l,m+\frac{1}{2}}^j + \bar{\phi}_{l,m-\frac{1}{2}}^j - 4 \widehat{\bar{\phi}}_{l,m}^j \right) \right] + \bar{T}_{l,m}^j, \end{aligned}$$

where

$$\bar{T}_{l,m}^j = \mathcal{O}(\tau^2 h^2 + \tau h^4 + h^6).$$

Now by the help of (2.22)-(2.26), (2.39), (2.40), (2.48), (2.56), (2.58)-(2.61), from (2.33) and (2.62), we obtain

$$\begin{aligned} \frac{\nu \tau h^2}{2} (\omega_{201} + \omega_{021}) &= \frac{h^2}{3} \left[\frac{3\tau}{2} (\omega_{002} + \alpha + \beta \omega_{001} + \gamma \omega_{101} + \eta \omega_{011}) \right. \\ &\left. + \frac{h^2}{12} (T_1 + T_2 - 8(\gamma T_3 + \eta T_4 + \beta T_5 + T_6)) \right] + \bar{T}_{l,m}^j. \tag{2.63} \end{aligned}$$

Finally, by the help of the relation (2.36), from (2.63), we get the local truncation error

$$\begin{aligned} \bar{T}_{l,m}^j &= \frac{-h^4}{12} \left[(1 - 16c_3) \beta \omega_{200} + (1 - 16c_4) \beta \omega_{020} - (1 + 16\nu a_3) \gamma \omega_{300} \right. \\ &+ \{1 - 16(a_3 \nu + 2a_4)\} \gamma \omega_{120} - (1 + 16\nu b_3) \eta \omega_{030} \\ &\left. + \{1 - 16(b_3 \nu + 2b_4)\} \eta \omega_{210} + (1 - 16d_3) \omega_{201} + (1 - 16d_4) \omega_{021} \right] \\ &+ \mathcal{O}(\tau^2 h^2 + \tau h^4 + h^6). \tag{2.64} \end{aligned}$$

For the proposed method (2.62) to be of $\mathcal{O}(\tau^2 + \tau h^2 + h^4)$, the coefficient of h^4 in (2.64) must be zero. Thus we obtain a set of equations

$$\begin{aligned} 1 - 16c_3 &= 0, & 1 - 16c_4 &= 0, \\ 1 + 16\nu a_3 &= 0, & 1 - 16(a_3 \nu + 2a_4) &= 0, \\ 1 + 16\nu b_3 &= 0, & 1 - 16(b_3 \nu + 2b_4) &= 0, \\ 1 - 16d_3 &= 0, & 1 - 16d_4 &= 0. \end{aligned} \tag{2.65}$$

On solving the system of equations (2.65), we obtain the values of parameters

$$a_3 = b_3 = -\frac{1}{16\nu}, \quad a_4 = b_4 = c_3 = c_4 = d_3 = d_4 = \frac{1}{16},$$

and the local truncation error becomes $\bar{T}_{l,m}^j = \mathcal{O}(\tau^2 h^2 + \tau h^4 + h^6)$.

Similarly, we can discretize the system of nonlinear parabolic PDEs with more than one dependent variables.

3. Two-level implicit schemes for time-dependent Burgers' and N-S equations in polar cylindrical coordinates

Consider the Burgers' equation in cylindrical polar coordinates with a forcing function

$$\frac{1}{Re} \left(u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u + u_{zz} \right) = u_t + u (u_r + u_z) + f(r, z, t),$$

$$0 < r, z < 1, \quad t > 0, \tag{3.1}$$

or, equivalently,

$$\nu (u_{rr} + u_{zz}) = u_t + D(r) u_r + E(r) u + u (u_r + u_z) + f(r, z, t),$$

$$0 < r, z < 1, \quad t > 0, \tag{3.2}$$

where $\nu = Re^{-1}$ is the coefficient of viscosity, $D(r) = -\frac{\nu}{r}$, $E(r) = \frac{\nu}{r^2}$ and $Re > 0$ is the Reynolds number. Proposed method (2.62) for Eq. (3.2), results into the following scheme:

$$\begin{aligned} & \nu \left(\delta_r^2 + \delta_z^2 + \frac{1}{6} \delta_r^2 \delta_z^2 \right) \bar{u}_{l,m}^j \\ = & \frac{h^2}{3} \bar{u}_{l+\frac{1}{2},m}^j + D_{l+\frac{1}{2}} \bar{u}_{r,l+\frac{1}{2},m}^j + E_{l+\frac{1}{2}} \bar{u}_{l+\frac{1}{2},m}^j + \bar{u}_{l+\frac{1}{2},m}^j \left(\bar{u}_{r,l+\frac{1}{2},m}^j + \bar{u}_{z,l+\frac{1}{2},m}^j \right) \\ & + \bar{f}_{l+\frac{1}{2},m}^j + \bar{u}_{l-\frac{1}{2},m}^j + D_{l-\frac{1}{2}} \bar{u}_{r,l-\frac{1}{2},m}^j + E_{l-\frac{1}{2}} \bar{u}_{l-\frac{1}{2},m}^j + \bar{u}_{l-\frac{1}{2},m}^j \left(\bar{u}_{r,l-\frac{1}{2},m}^j + \bar{u}_{z,l-\frac{1}{2},m}^j \right) \\ & + \bar{f}_{l-\frac{1}{2},m}^j + \bar{u}_{l,m+\frac{1}{2}}^j + D_l \bar{u}_{r,l,m+\frac{1}{2}}^j + E_l \bar{u}_{l,m+\frac{1}{2}}^j + \bar{u}_{l,m+\frac{1}{2}}^j \left(\bar{u}_{r,l,m+\frac{1}{2}}^j + \bar{u}_{z,l,m+\frac{1}{2}}^j \right) \\ & + \bar{f}_{l,m+\frac{1}{2}}^j + \bar{u}_{l,m-\frac{1}{2}}^j + D_l \bar{u}_{r,l,m-\frac{1}{2}}^j + E_l \bar{u}_{l,m-\frac{1}{2}}^j + \bar{u}_{l,m-\frac{1}{2}}^j \left(\bar{u}_{r,l,m-\frac{1}{2}}^j + \bar{u}_{z,l,m-\frac{1}{2}}^j \right) \\ & + \bar{f}_{l,m-\frac{1}{2}}^j + 3 \left\{ \bar{u}_{l,m}^j + D_l \bar{u}_{r,l,m}^j + E_l \bar{u}_{l,m}^j + \bar{u}_{l,m}^j \left(\bar{u}_{r,l,m}^j + \bar{u}_{z,l,m}^j \right) + \bar{f}_{l,m}^j \right\} \\ & - 4 \left\{ \hat{u}_{l,m}^j + D_l \hat{u}_{r,l,m}^j + E_l \hat{u}_{l,m}^j + \hat{u}_{l,m}^j \left(\hat{u}_{r,l,m}^j + \hat{u}_{z,l,m}^j \right) + \bar{f}_{l,m}^j \right\}, \end{aligned} \tag{3.3}$$

where the required approximations for the scheme (3.3) are already defined in Section 2 in terms of ω .

Next, we consider the model N-S equations in cylindrical polar coordinates

$$Re^{-1} \left(u_{rr} + \frac{1}{r} u_r + u_{zz} - \frac{1}{r^2} u \right) = u_t + uu_r + vu_z + f(r, z, t),$$

$$0 < r, z < 1, \quad t > 0, \tag{3.4}$$

$$Re^{-1} \left(v_{rr} + \frac{1}{r} v_r + v_{zz} \right) = v_t + uv_r + vv_z + g(r, z, t),$$

$$0 < r, z < 1, \quad t > 0, \tag{3.5}$$

or, equivalently

$$\begin{aligned} \nu (u_{rr} + u_{zz}) &= u_t + D(r) u_r + E(r) u + uu_r + vv_z + f(r, z, t), \\ 0 < r, z < 1, \quad t > 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \nu (v_{rr} + v_{zz}) &= v_t + D(r) v_r + uv_r + vv_z + g(r, z, t), \\ 0 < r, z < 1, \quad t > 0, \end{aligned} \tag{3.7}$$

where u, v are the velocity components, $Re > 0$ represents the Reynolds number, $\nu = Re^{-1}$ is the coefficient of viscosity, $D(r) = -\frac{\nu}{r}$, $E(r) = \frac{\nu}{r^2}$.

The continuity equation for incompressible flow in cylindrical polar coordinates is given by

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0. \tag{3.8}$$

The continuity equation (3.8) is a first order PDE without a time derivative term. The maximum order of accuracy of numerical method (in compact form) for continuity equation is two, thus cannot be fitted in our method (2.62). Hence for computation, we have chosen exact solutions in such a manner which satisfied the continuity equation. Assume that the pressure gradient functions $f(r, z, t), g(r, z, t)$ are known so that the system of equations (3.4)-(3.5) or (3.6)-(3.7) can be solved uniquely for the velocity components u and v .

Applying the method (2.62) in coupled form to the Eqs. (3.6)-(3.7), we get

$$\begin{aligned} &\nu \left(\delta_r^2 + \delta_z^2 + \frac{1}{6} \delta_r^2 \delta_z^2 \right) \bar{u}_{l,m}^j \\ &= \frac{h^2}{3} \left[\bar{u}_{l+\frac{1}{2},m}^j + D_{l+\frac{1}{2}} \bar{u}_{r,l+\frac{1}{2},m}^j + E_{l+\frac{1}{2}} \bar{u}_{l+\frac{1}{2},m}^j + \bar{u}_{l+\frac{1}{2},m}^j \bar{u}_{r,l+\frac{1}{2},m}^j \right. \\ &\quad + \bar{v}_{l+\frac{1}{2},m}^j \bar{u}_{z,l+\frac{1}{2},m}^j + \bar{f}_{l+\frac{1}{2},m}^j + \bar{u}_{l-\frac{1}{2},m}^j + D_{l-\frac{1}{2}} \bar{u}_{r,l-\frac{1}{2},m}^j \\ &\quad + E_{l-\frac{1}{2}} \bar{u}_{l-\frac{1}{2},m}^j + \bar{u}_{l-\frac{1}{2},m}^j \bar{u}_{r,l-\frac{1}{2},m}^j + \bar{v}_{l-\frac{1}{2},m}^j \bar{u}_{z,l-\frac{1}{2},m}^j + \bar{f}_{l-\frac{1}{2},m}^j \\ &\quad + \bar{u}_{l,m+\frac{1}{2}}^j + D_l \bar{u}_{r,l,m+\frac{1}{2}}^j + E_l \bar{u}_{l,m+\frac{1}{2}}^j + \bar{u}_{l,m+\frac{1}{2}}^j \bar{u}_{r,l,m+\frac{1}{2}}^j \\ &\quad + \bar{v}_{l,m+\frac{1}{2}}^j \bar{u}_{z,l,m+\frac{1}{2}}^j + \bar{f}_{l,m+\frac{1}{2}}^j + \bar{u}_{l,m-\frac{1}{2}}^j + D_l \bar{u}_{r,l,m-\frac{1}{2}}^j \\ &\quad + E_l \bar{u}_{l,m-\frac{1}{2}}^j + \bar{u}_{l,m-\frac{1}{2}}^j \bar{u}_{r,l,m-\frac{1}{2}}^j + \bar{v}_{l,m-\frac{1}{2}}^j \bar{u}_{z,l,m-\frac{1}{2}}^j + \bar{f}_{l,m-\frac{1}{2}}^j \\ &\quad + 3 \left(\bar{u}_{l,m}^j + D_l \bar{u}_{r,l,m}^j + E_l \bar{u}_{l,m}^j + \bar{u}_{l,m}^j \bar{u}_{r,l,m}^j + \bar{v}_{l,m}^j \bar{u}_{z,l,m}^j + \bar{f}_{l,m}^j \right) \\ &\quad \left. - 4 \left(\hat{u}_{l,m}^j + D_l \hat{u}_{r,l,m}^j + E_l \hat{u}_{l,m}^j + \hat{u}_{l,m}^j \hat{u}_{r,l,m}^j + \hat{v}_{l,m}^j \hat{u}_{z,l,m}^j + \hat{f}_{l,m}^j \right) \right], \tag{3.9} \\ &\nu \left(\delta_r^2 + \delta_z^2 + \frac{1}{6} \delta_r^2 \delta_z^2 \right) \bar{v}_{l,m}^j \end{aligned}$$

$$\begin{aligned}
 &= \frac{h^2}{3} \left[\bar{v}_{l+\frac{1}{2},m}^j + D_{l+\frac{1}{2}} \bar{v}_{r_{l+\frac{1}{2},m}}^j + \bar{u}_{l+\frac{1}{2},m}^j \bar{v}_{r_{l+\frac{1}{2},m}}^j + \bar{v}_{l+\frac{1}{2},m}^j \bar{v}_{z_{l+\frac{1}{2},m}}^j + \bar{g}_{l+\frac{1}{2},m}^j \right. \\
 &\quad + \bar{v}_{l-\frac{1}{2},m}^j + D_{l-\frac{1}{2}} \bar{v}_{r_{l-\frac{1}{2},m}}^j + \bar{u}_{l-\frac{1}{2},m}^j \bar{v}_{r_{l-\frac{1}{2},m}}^j + \bar{v}_{l-\frac{1}{2},m}^j \bar{v}_{z_{l-\frac{1}{2},m}}^j + \bar{g}_{l-\frac{1}{2},m}^j \\
 &\quad + \bar{v}_{l,m+\frac{1}{2}}^j + D_l \bar{v}_{r_{l,m+\frac{1}{2}}}^j + \bar{u}_{l,m+\frac{1}{2}}^j \bar{v}_{r_{l,m+\frac{1}{2}}}^j + \bar{v}_{l,m+\frac{1}{2}}^j \bar{v}_{z_{l,m+\frac{1}{2}}}^j + \bar{g}_{l,m+\frac{1}{2}}^j \\
 &\quad + \bar{v}_{l,m-\frac{1}{2}}^j + D_l \bar{v}_{r_{l,m-\frac{1}{2}}}^j + \bar{u}_{l,m-\frac{1}{2}}^j \bar{v}_{r_{l,m-\frac{1}{2}}}^j + \bar{v}_{l,m-\frac{1}{2}}^j \bar{v}_{z_{l,m-\frac{1}{2}}}^j + \bar{g}_{l,m-\frac{1}{2}}^j \\
 &\quad + 3 \left(\bar{v}_{l,m}^j + D_l \bar{v}_{r_{l,m}}^j + \bar{u}_{l,m}^j \bar{v}_{r_{l,m}}^j + \bar{v}_{l,m}^j \bar{v}_{z_{l,m}}^j + \bar{g}_{l,m}^j \right) \\
 &\quad \left. - 4 \left(\widehat{\bar{v}}_{l,m}^j + D_l \widehat{\bar{v}}_{r_{l,m}}^j + \widehat{\bar{u}}_{l,m}^j \widehat{\bar{v}}_{r_{l,m}}^j + \widehat{\bar{v}}_{l,m}^j \widehat{\bar{v}}_{z_{l,m}}^j + \bar{g}_{l,m}^j \right) \right], \tag{3.10}
 \end{aligned}$$

where the approximations for u and v associated with (3.8) and (3.9) are defined in Section 2 in terms of ω .

4. Stability consideration

For stability consider the following singular equation, which is a linearized form of the Burgers' equation (3.2) and is given by

$$v(\omega_{rr} + \omega_{zz}) = \omega_t + D(r)\omega_r, \quad 0 < r, z < 1, \quad t > 0, \tag{4.1}$$

where $D(r) = \frac{1}{r}$. Applying the proposed method (2.62) to the Eq (4.1), we get the scheme in difference operator form

$$\begin{aligned}
 &\left[1 + \frac{1}{12} (1 - 6v\lambda + \lambda P_1) \delta_r^2 + \frac{1}{12} (1 - 6v\lambda) \delta_z^2 - \frac{1}{12} \left(\frac{hD_l}{2v} - \lambda P_2 \right) (2\mu_r \delta_r) \right. \\
 &\quad \left. + \frac{h\lambda D_l}{24} (2\delta_z^2 \mu_r \delta_r) - \frac{v\lambda \delta_r^2 \delta_z^2}{12} \right] \omega_{l,m}^{j+1} \\
 &= \left[1 + \frac{1}{12} (1 + 6v\lambda - \lambda P_1) \delta_r^2 + \frac{1}{12} (1 + 6v\lambda) \delta_z^2 - \frac{1}{12} \left(\frac{hD_l}{2v} + \lambda P_2 \right) (2\mu_r \delta_r) \right. \\
 &\quad \left. - \frac{h\lambda D_l}{24} (2\delta_z^2 \mu_r \delta_r) + \frac{v\lambda \delta_r^2 \delta_z^2}{12} \right] \omega_{l,m}^j, \tag{4.2}
 \end{aligned}$$

where

$$D_l = D(r_l) = \frac{1}{r_l}, \quad (D_r)_l = D_r(r_l) = \frac{-1}{(r_l)^2}, \quad (D_{rr})_l = D_{rr}(r_l) = \frac{2}{(r_l)^3},$$

$$P_1 = h^2 \left((D_r)_l - \frac{(D_l)^2}{2v} \right), \quad P_2 = h \left(3D_l + \frac{h^2}{4} \left\{ (D_{rr})_l - \frac{D_l(D_r)_l}{v} \right\} \right),$$

$$\delta_r^2 \omega_{l,m}^{j+1} = \omega_{l+1,m}^{j+1} - 2\omega_{l,m}^{j+1} + \omega_{l-1,m}^{j+1}, \quad \delta_z^2 \omega_{l,m}^{j+1} = \omega_{l,m+1}^{j+1} - 2\omega_{l,m}^{j+1} + \omega_{l,m-1}^{j+1},$$

$$(2\mu_r \delta_r) \omega_{l,m}^{j+1} = \omega_{l+1,m}^{j+1} - \omega_{l-1,m}^{j+1}, \quad (2\mu_r \delta_r) \omega_{l,m}^j = \omega_{l+1,m}^j - \omega_{l-1,m}^j,$$

$$\begin{aligned}
 (2\delta_z^2 \mu_r \delta_r) \omega_{l,m}^{j+1} &= \omega_{l+1,m+1}^{j+1} + \omega_{l+1,m-1}^{j+1} - \omega_{l-1,m+1}^{j+1} - \omega_{l-1,m-1}^{j+1} - 2 \left(\omega_{l+1,m}^{j+1} - \omega_{l-1,m}^{j+1} \right), \\
 \delta_r^2 \delta_z^2 \omega_{l,m}^{j+1} &= \omega_{l+1,m+1}^{j+1} + \omega_{l+1,m-1}^{j+1} + \omega_{l-1,m+1}^{j+1} + \omega_{l-1,m-1}^{j+1} \\
 &\quad - 2 \left(\omega_{l+1,m}^{j+1} + \omega_{l-1,m}^{j+1} + \omega_{l,m+1}^{j+1} + \omega_{l,m-1}^{j+1} \right) + 4\omega_{l,m}^{j+1}, \\
 \delta_r^2 \omega_{l,m}^j &= \omega_{l+1,m}^j - 2\omega_{l,m}^j + \omega_{l-1,m}^j, \quad \delta_z^2 \omega_{l,m}^j = \omega_{l,m+1}^j - 2\omega_{l,m}^j + \omega_{l,m-1}^j, \\
 (2\delta_z^2 \mu_r \delta_r) \omega_{l,m}^j &= \omega_{l+1,m+1}^j + \omega_{l+1,m-1}^j - \omega_{l-1,m+1}^j - \omega_{l-1,m-1}^j - 2 \left(\omega_{l+1,m}^j - \omega_{l-1,m}^j \right), \\
 \delta_r^2 \delta_z^2 \omega_{l,m}^j &= \omega_{l+1,m+1}^j + \omega_{l+1,m-1}^j + \omega_{l-1,m+1}^j + \omega_{l-1,m-1}^j \\
 &\quad - 2 \left(\omega_{l+1,m}^j + \omega_{l-1,m}^j + \omega_{l,m+1}^j + \omega_{l,m-1}^j \right) + 4\omega_{l,m}^j.
 \end{aligned}$$

From the above notations, it is clear that the left-hand and right-hand sides of (4.2) depend upon compact 9-spatial grid points at $(j + 1)$ -th and j -th time levels. Thus scheme (4.2) is a two-level implicit scheme, and each advanced time-level contains 9-spatial grid points. The scheme (4.2) can be written as a 9-diagonal sparse matrix form, which can be solved using a suitable iterative method at each advanced time-level. In order to ease the computation, we use the alternating direction implicit (ADI) [20] method (direct method) described as follows:

We can write Eq. (4.2) in product form as

$$[R_1][Z_1]\omega_{l,m}^{j+1} = [R_2][Z_2]\omega_{l,m}^j, \tag{4.3}$$

where

$$\begin{aligned}
 R_1 &= \left[1 + \frac{1}{12} (1 - 6v\lambda + \lambda P_1) \delta_r^2 - \frac{1}{12} \left(\frac{hD_l}{2v} - \lambda P_2 \right) (2\mu_r \delta_r) \right], \\
 R_2 &= \left[1 + \frac{1}{12} (1 + 6v\lambda - \lambda P_1) \delta_r^2 - \frac{1}{12} \left(\frac{hD_l}{2v} + \lambda P_2 \right) (2\mu_r \delta_r) \right], \\
 Z_1 &= \left[1 + \frac{1}{12} (1 - 6v\lambda) \delta_z^2 \right], \quad Z_2 = \left[1 + \frac{1}{12} (1 + 6v\lambda) \delta_z^2 \right].
 \end{aligned}$$

The additional term added in (4.3) is of higher order, which is affectless to the accuracy of the scheme. We split the scheme (4.3) and in ADI form can be written as:

$$[Z_1]\omega_{l,m}^* = [R_2][Z_2]\omega_{l,m}^j, \tag{4.4}$$

$$[R_1]\omega_{l,m}^{j+1} = \omega_{l,m}^*, \tag{4.5}$$

where $\omega_{l,m}^*$ is any mock variable and the required boundary conditions of mock variable for solving (4.4) can be obtained from (4.5). The left-hand sides of (4.4) and (4.5) are tri-diagonal matrices, thus can be solved using a tri-diagonal solver.

For stability of (4.3), we assume that an error $\varepsilon_{l,m}^j = \xi^j e^{i\theta_1 l h} e^{i\theta_2 m h}$ exists at each internal mesh point (r_l, z_m, t_j) , where $i = \sqrt{-1}$, θ_1 and θ_2 are phase angles, then the

characteristic equation of (4.3) may be written as

$$\begin{aligned} \xi = & \left[1 - \frac{1}{3} (1 + 6v\lambda - \lambda P_1) \sin^2 \left(\frac{\theta_1}{2} \right) - \frac{i}{6} \left(\frac{hD_l}{2v} + \lambda P_2 \right) \sin(\theta_1) \right] \\ & \times \left[1 - \frac{1}{3} (1 + 6v\lambda) \sin^2 \left(\frac{\theta_2}{2} \right) \right] \\ & \times \left(\left[1 - \frac{1}{3} (1 - 6v\lambda + \lambda P_1) \sin^2 \left(\frac{\theta_1}{2} \right) - \frac{i}{6} \left(\frac{hD_l}{2v} - \lambda P_2 \right) \sin(\theta_1) \right] \right. \\ & \left. \times \left[1 - \frac{1}{3} (1 - 6v\lambda) \sin^2 \left(\frac{\theta_2}{2} \right) \right] \right)^{-1} = \xi_1 \cdot \xi_2, \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} \xi_1 = & \left[1 - \frac{1}{3} (1 + 6v\lambda - \lambda P_1) \sin^2 \left(\frac{\theta_1}{2} \right) - \frac{i}{6} \left(\frac{hD_l}{2v} + \lambda P_2 \right) \sin(\theta_1) \right] \\ & \times \left[1 - \frac{1}{3} (1 - 6v\lambda + \lambda P_1) \sin^2 \left(\frac{\theta_1}{2} \right) - \frac{i}{6} \left(\frac{hD_l}{2v} - \lambda P_2 \right) \sin(\theta_1) \right]^{-1}, \\ \xi_2 = & \left[1 - \frac{1}{3} (1 + 6v\lambda) \sin^2 \left(\frac{\theta_2}{2} \right) \right] \left[1 - \frac{1}{3} (1 - 6v\lambda) \sin^2 \left(\frac{\theta_2}{2} \right) \right]^{-1}. \end{aligned}$$

For stability, we require that ξ should be bounded by 1. It is observed from [20] that $|\xi_1|^2 \leq 1$. In a similar way, we can show that $|\xi_2|^2 \leq 1$. Therefore $|\xi|^2 \leq 1$ and the linear scheme for the singular problem presented by (4.3), is unconditionally stable.

5. Numerical results

In order to verify the efficiency and accuracy of the proposed method, we solve the following one linear and three nonlinear problems in the r - z plane, whose exact solutions are given as a test procedure. The right-hand side homogeneous functions, initial and boundary values can be obtained from the exact solutions. ADI method is used to solve linear difference equation, and Newton's nonlinear block iteration method [8, 13] is used to solve the system of nonlinear difference equations. Absolute error tolerance for the termination of iterations is kept $\leq 10^{-15}$ and the initial guess is assumed trivial i.e. $u = 0$. The present results are compared with those computed by using the method discussed in [20]. All the computations are performed by using MATLAB code.

Problem 1: (Heat equation in cylindrical polar coordinates)

$$\frac{\partial^2 u}{\partial r^2} + \frac{\alpha}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} + f(r, t), \quad 0 < r, z < 1, \quad t > 0. \quad (5.1)$$

For $\alpha = 1$, the Eq. (5.1) is called heat equation in cylindrical polar coordinates. The exact solution is given by $u(r, z, t) = \exp(-at) \cosh r \cosh z$. For a fixed value of $\lambda = 3.2$,

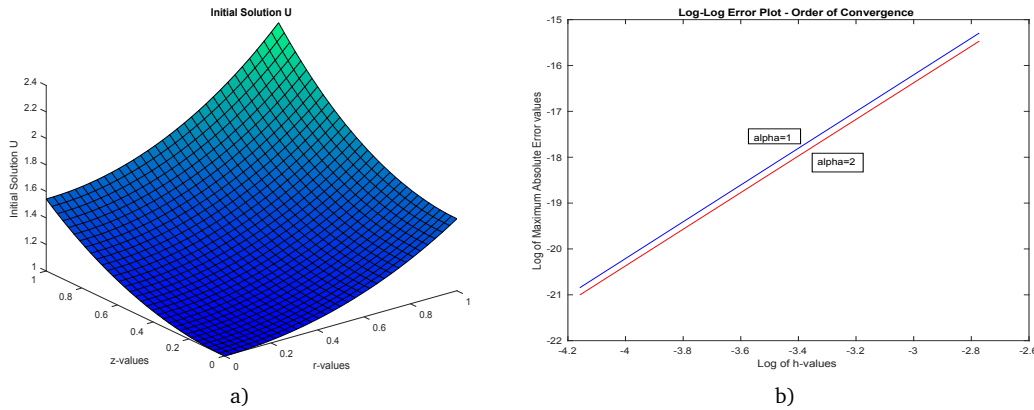


Figure 1: Problem 1: a) The initial solution for u (at $t = 0$), $h = 1/32$; b) Log-Log error plot for u .

Table 1: Problem 1: The maximum absolute errors at $t = 1$.

h	Proposed Method (2.62)		Results using the method [20]	
	$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
$\frac{1}{16}$	1.9125e-07	2.2805e-07	3.9472e-07	4.8427e-07
CPU time in secs	(0.282)	(0.293)	(0.312)	(0.322)
$\frac{1}{32}$	1.2036e-08	1.4218e-08	2.2857e-08	3.0286e-08
CPU time in secs	(7.142)	(7.212)	(8.728)	(8.912)
$\frac{1}{64}$	7.5612e-10	8.8542e-10	1.4141e-09	1.8931e-09
CPU time in secs	(32.112)	(32.314)	(35.692)	(36.454)

the maximum absolute errors (MAE) for u are tabulated in Table 1 at $t = 1$ for $\alpha = 1$ and $\alpha = 2$. The graphical illustration for initial solution (at $t = 0$) is shown in Fig. 1(a), and log-log error plot is shown in Fig. 1(b).

Problem 2: (Viscous unsteady Burgers' equation in $r-z$ plane)

We solve the nonlinear differential equation (3.1) in the $r-z$ plane ($0 < r < 1, 0 < z < 1, t > 0$), whose exact solution is given by $u(r, z, t) = \exp(-2\nu\pi^2 t)\pi^2 r^2 \sin(\pi z)$. For a fixed value of $\lambda = 3.2$, the MAEs for are tabulated in Table 2 at $t = 5$ for $Re = \nu^{-1} = 10$ and $Re = \nu^{-1} = 100$. The graphical illustration for initial solution is portrayed in Fig. 2(a), and log-log error plot is portrayed in Fig. 2(b).

Problem 3: (Time-dependent Navier-Stokes equations of motion in $r-z$ plane)

We solve the N-S equations of motion (3.4)-(3.5) in $r-z$ plane ($0 < r < 1, 0 < z < 1, t > 0$), whose exact solutions are given by

$$u(r, z, t) = \exp\left(-\frac{t}{Re}\right) r^3 \sinh(z), \quad v(r, z, t) = -4\exp\left(-\frac{t}{Re}\right) r^2 \cosh(z).$$

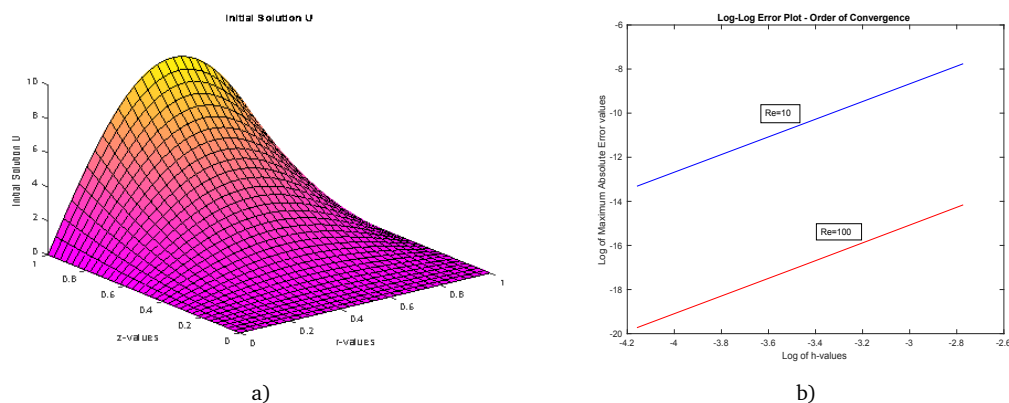


Figure 2: Problem 2: a) The initial solution for u (at $t = 0$), $h = 1/32$; b) Log-Log error plot for u .

Table 2: Problem 2: The maximum absolute errors at $t = 5$.

h	Proposed Method (2.62)		Results using the method [20]	
	$Re = 10$	$Re = 100$	$Re = 10$	$Re = 100$
$\frac{1}{16}$	7.1003e-07	4.2667e-04	2.4132e-06	2.4507e-03
CPU time in secs	(0.717)	(0.856)	(0.828)	(0.992)
$\frac{1}{32}$	4.3354e-08	2.6484e-05	1.7049e-07	1.5569e-04
CPU time in secs	(10.554)	(12.638)	(12.169)	(15.284)
$\frac{1}{64}$	2.6984e-09	1.6542e-06	1.1313e-08	9.8511e-06
CPU time in secs	(88.129)	(98.672)	(96.438)	(106.531)

Table 3: Problem 3: The maximum absolute errors at $t = 1$.

h		Proposed Method (2.62)		Results using the method [20]	
		$Re = 10$	$Re = 100$	$Re = 10$	$Re = 100$
$\frac{1}{16}$	u	5.1123e-07	4.6032e-04	2.8068e-06	6.3618e-03
	v	4.2570e-07	3.8555e-04	2.1001e-06	5.4629e-03
CPU time in secs		(3.982)	(5.926)	(4.616)	(6.711)
$\frac{1}{32}$	u	3.1815e-08	2.8586e-05	1.7518e-07	3.9854e-04
	v	2.6606e-08	2.3989e-05	1.3124e-07	2.8257e-04
CPU time in secs		(66.138)	(71.660)	(72.112)	(84.184)
$\frac{1}{64}$	u	1.9884e-09	1.7946e-06	1.1012e-08	2.4816e-05
	v	1.6596e-09	1.4996e-06	8.2125e-09	1.7614e-05
CPU time in secs		(466.198)	(496.164)	(492.314)	(528.662)

For several values of Re and fixed value of $\lambda = 3.2$, the MAEs corresponding to unknowns u and v are tabulated in Table 3 at $t = 1$. The initial solutions for u, v are plotted in Figs. 3(a) and 3(b), and log-log error plots are shown in Figs. 3(c) and 3(d).

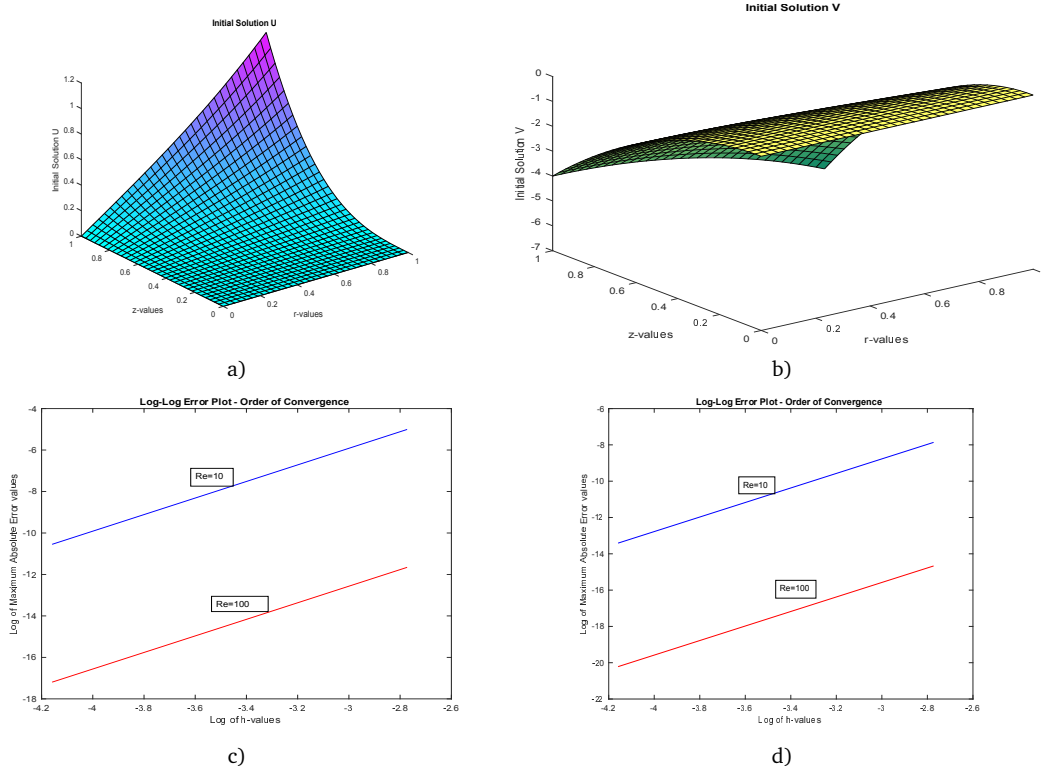


Figure 3: Problem 3: a) The initial solution for u (at $t = 0$), $h = 1/32$; b) The initial solution for v (at $t = 0$), $h = 1/32$; c) Log-Log error plot for u ; d) Log-Log error plot for v .

Problem 4: (Taylor-Vortex Problem in r - z plane)

We solve the N-S equations of motion (3.4)-(3.5) in r - z plane ($0 < r < 1, 0 < z < 1, t > 0$), whose exact solutions are given by

$$u(r, z, t) = \exp\left(-\frac{\pi^2 N^2 t}{Re}\right) \pi N^3 r^3 \sin(\pi N z),$$

$$v(r, z, t) = 4 \exp\left(-\frac{\pi^2 N^2 t}{Re}\right) N^2 r^2 \cos(\pi N z),$$

where N is the number of vortices.

For several values of Re and fixed value of $\lambda = 3.2$ and $N = 4$, the MAEs corresponding to u and v are tabulated in Table 4 at $t = 1$. The initial solutions for u, v are graphed in Figs. 4(a) and 4(b), and log-log error plots are given in Figs. 4(c) and 4(d).

The number of errors = N^2 = Number of internal grid points at j -th time level. The order of convergence is computed as

$$C_R = \log\left(\frac{e_{h_1}}{e_{h_2}}\right) \left(\log\left(\frac{h_1}{h_2}\right)\right)^{-1},$$

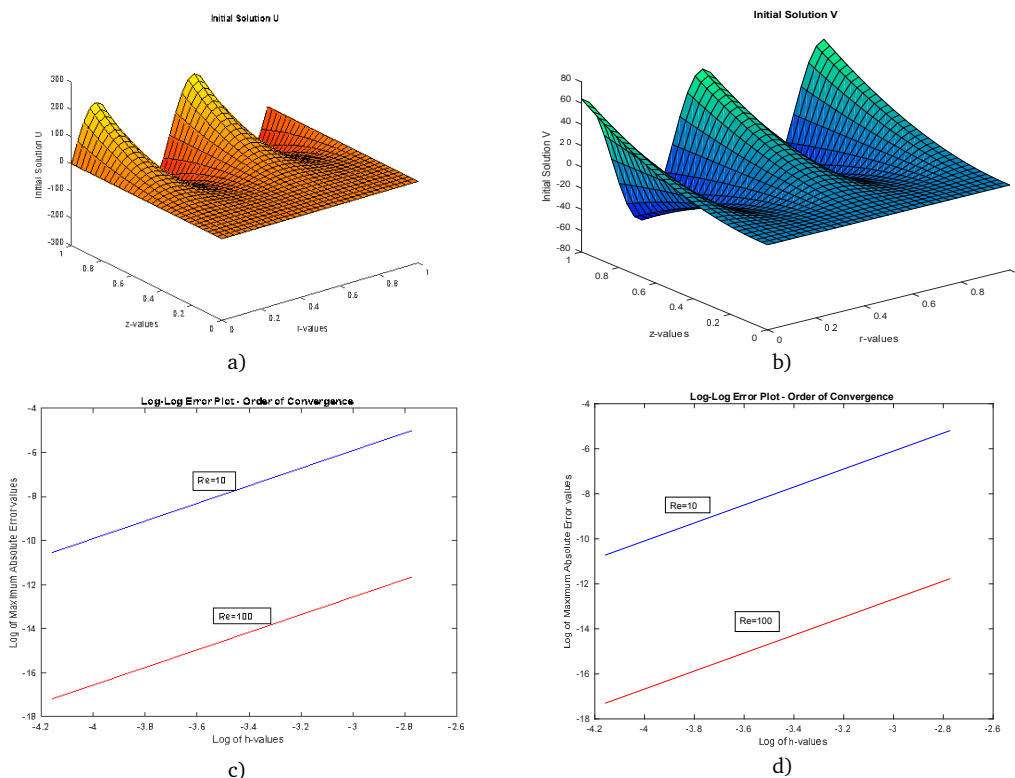


Figure 4: Problem 4: a) The initial solution for u (at $t = 0$), $h = 1/32$; b) The initial solution for v (at $t = 0$), $h = 1/32$; c) Log-Log error plot for u ; d) Log-Log error plot for v .

Table 4: Problem 4: The maximum absolute errors at $t = 1$ for $N = 4$.

h		Proposed Method (2.62)		Results using the method [20]	
		$Re = 10$	$Re = 100$	$Re = 10$	$Re = 100$
$\frac{1}{16}$	u	8.6632e-06	6.6937e-03	4.2772e-05	2.4607e-02
	v	7.7278e-06	5.5557e-03	3.8719e-05	1.8252e-02
CPU time in secs		(5.549)	(8.125)	(6.188)	(9.782)
$\frac{1}{32}$	u	5.4252e-07	4.1872e-04	2.6808e-06	1.5469e-03
	v	4.8562e-07	3.4878e-04	2.4274e-06	1.1453e-03
CPU time in secs		(86.728)	(111.172)	(98.468)	(124.884)
$\frac{1}{64}$	u	3.4097e-08	2.6248e-05	1.6798e-07	9.6911e-05
	v	3.0445e-08	2.1948e-05	1.5215e-07	7.1682e-05
CPU time in secs		(544.841)	(632.663)	(578.116)	(684.658)

where e_{h_1} and e_{h_2} are maximum absolute errors for two different uniform mesh sizes h_1 and h_2 respectively. For the order of convergence, we have chosen two small values of h , that is, $h_1 = \frac{1}{32}$, $h_2 = \frac{1}{64}$ and the corresponding maximum absolute errors e_{h_1} and e_{h_2} and computed the order of convergence in Table 5. It is automatically understood that for $h_1 = \frac{1}{32}$ and $h_2 = \frac{1}{64}$, the number of errors are 31^2 and 63^2 , respectively. Hence,

Table 5: The rate of convergence: $h_1 = 1/32, h_2 = 1/64$.

Problem	Parameters, if any	Order of convergence	
1	$\alpha = 1$	3.99	
	$\alpha = 2$	4.00	
2	$Re = 10$	4.00	
	$Re = 100$	4.00	
3	$Re = 10$	for u	4.00
		for v	4.00
	$Re = 100$	for u	3.99
		for v	3.99
4	$Re = 10$	for u	3.99
		for v	3.99
	$Re = 100$	for u	3.99
		for v	3.99

there is no need of mentioning number of errors in Table 5. This formula is widely used by the researchers for the computation of order of convergence. Different values of C_R for all the problems are reported in Table 5.

6. Summary

The existing two-level compact implicit method [20] is temporally order two and spatially order four accurate for the numerical solution of nonlinear 2D parabolic equations in polar coordinates. While the method [20] is directly applicable to solve 2D parabolic equations irrespective of coordinates, it suffers from a drawback that it involves a lot of algebra for computation. In this paper, using less algebra, we formulated a new implicit scheme in exponential form for 2D nonlinear parabolic equations in polar cylindrical r - z plane, and hence for the solution of Burgers' and N-S equations of motion in the r - z plane. The proposed scheme is of order two in time and four in space and the computational stencil requires only nine points at the advanced time level. The stability analysis for the heat equation in polar cylindrical coordinates was discussed and found to be unconditionally stable. Accurate and efficient computation of four benchmark problems illustrated the effectiveness of the proposed method. The accuracy and efficiency of the numerical solutions can be judged by the results obtained in all the three cases on relatively coarser grids, which are in transcendent agreement with analytical and available numerical results. Construction of high order method is possible, only when the solution is sufficiently smooth and the solution domain is regular, that is, no discontinuities in the solution domain. The current scheme is not applicable to the problems with discontinuities. The proposed method has the potential to be extended to 3D problems on which, we are presently working.

Acknowledgements

This research was partly supported by Chinese Academy of Sciences under INSA-CAS bilateral exchange programme-2019, and partly supported by University Grant Commission, New Delhi, India under the Ref. No 1087, and L. Yuan thanks the support of NSFC (91641107, 91852116, 12071470).

The authors thank the reviewers for their valuable suggestions, which substantially enhanced the standard of the paper.

References

- [1] Z. Z. BAI, Y. H. RAN AND L. YUAN, *On approximated ILU and UGS preconditioning methods for linearized discretized steady incompressible Navier-Stokes equations*, Numer. Algorithms, 65 (2014) 43–68.
- [2] A. BALZANO, *Mosquito: An efficient finite difference scheme for numerical simulation of 2D advection*, Int. J. Numer. Meth. Fluids, 31 (1999) 481–496.
- [3] A. J. CHORIN, *Numerical solution of the Navier-Stokes equations*, Math. Comp., 22 (1968) 747–762.
- [4] W. DAI AND R. NASSAR, *Compact ADI method for solving parabolic differential equations*, Numer. Meth. Part. Diff. Eqn., 18 (2002) 129–142.
- [5] H. EREN, *Numerical study of unsteady interacting compressible jet flows using finite difference method for small Reynolds number*, Appl. Math. Comput., 172 (2006) 876–891.
- [6] E. ERTURK AND C. GOKCOL, *Fourth order compact formulation of the Navier-Stokes equations and driven cavity flow at higher Reynolds number*, Int. J. Numer. Meth. Fluids, 50 (2006) 421–436.
- [7] C. A. J. FLETCHER, *Generating exact solutions of the two-dimensional Burgers' equations*, Int. J. Numer. Meth. Fluids, 19 (1994) 369–375.
- [8] L. A. HAGEMAN AND D. M. YOUNG, *Applied Iterative Methods*, Dover Publication, 2004.
- [9] S. R. K. IYENGAR AND R. MANOHAR, *High order difference methods for heat equation in polar cylindrical coordinates*, J. Comput. Phys., 77 (1988) 425–438.
- [10] M. K. JAIN, R. K. JAIN AND R. K. MOHANTY, *A fourth order difference method for the one dimensional general quasi-linear parabolic partial differential equation*, Numer. Meth. Part. Diff. Eqns., 6 (1990) 311–319.
- [11] J. C. KALITA, D. C. DALAL AND A. K. DASS, *A class of higher order compact schemes for the unsteady two-dimensional convection–diffusion equation with variable convection coefficients*, Int. J. Numer. Meth. Fluids, 38 (2002) 1111–1131.
- [12] S. KARAA AND J. ZHANG, *High order ADI method for solving unsteady convection–diffusion problems*, J. Comput. Phys., 198 (2004) 1–9.
- [13] C. T. KELLY, *Iterative Methods for Linear and Nonlinear Equations*, SIAM Publication, 1995.
- [14] M. LI, T. TANG AND B. FORNBERG, *A compact fourth-order finite difference scheme for the steady-state incompressible Navier-Stokes equations*, Int. J. Numer. Methods Fluids, 20 (1995) 1137–1151.
- [15] R. K. MOHANTY, W. DAI AND D. LIU, *Operator compact method of accuracy two in time and four in space for the solution of time-dependent Burgers-Huxley equation*, Numer. Algorithms., 70 (2015) 591–605.

- [16] R. K. MOHANTY AND M. K. JAIN, *High accuracy difference schemes for the system of two space non-linear parabolic differential equations with mixed derivatives and variable coefficients*, J. Comput. Appl. Math., 70 (1996) 15–32.
- [17] R. K. MOHANTY, G. MANCHANDA AND A. KHAN, *Operator compact exponential approximation for the solution of the system of 2D second order quasi-linear elliptic partial differential equations*, Adv. Difference Equ., 2019 (2019) ID: 47.
- [18] R. K. MOHANTY, G. MANCHANDA AND A. KHAN, *Compact half step approximation in exponential form for 2D second order quasi-linear elliptic partial differential equations*, J. Differ. Equ. Appl., 25 (2019) 716–749.
- [19] R. K. MOHANTY, G. MANCHANDA, A. KHAN AND G. KHURANA, *A new high accuracy method in exponential form based on off-step discretization for non-linear two point boundary value problems*, J. Differ. Equ. Appl., 26 (2020) 171–202.
- [20] R. K. MOHANTY AND N. SETIA, *A new high accuracy two-level implicit off-step discretization for the system of two space dimensional quasi-linear parabolic partial differential equations*, Appl. Math. Comput., 219 (2012) 2680–2697.
- [21] R. K. MOHANTY, L. YUAN AND D. SHARMA, *A new compact scheme in exponential form for two-dimensional time-dependent Burgers' and Navier-Stokes equations*, East Asian J. Appl. Math., 10 (2020) 437–454.
- [22] J. M. C. PEREIR, M. H. KOBAYASHI AND J. C. F. PEREIRA, *A fourth order accurate finite volume compact method for the incompressible Navier-Stokes solutions*, J. Comput. Phys., 167 (2001) 217–243.
- [23] A. SHAH, H. GUO AND L. YUAN, *A third order upwind compact scheme on curvilinear meshes for the incompressible Navier-Stokes equations*, Commun. Comput. Phys., 5 (2009) 712–729.
- [24] P. N. SHANKAR AND M. D. DESHPANDE, *Fluid mechanics in the driven cavity*, Annu. Rev. Fluid Mech., 32 (2000) 93–136.
- [25] I. YAVNEH, *Analysis of a fourth-order compact scheme for convection diffusion*, J. Comput. Phys., 133 (1997) 361–364.