

A New Compact Scheme in Exponential Form for Two-Dimensional Time-Dependent Burgers' and Navier-Stokes Equations

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Abstract. A new compact implicit exponential scheme for Burgers' and Navier-Stokes equation is developed. The method has fourth order accuracy in space and second order accuracy in time. It uses only two time levels for computation and requires nine grid points at each time level. The stability of the method is proven for linearised Burgers' equation. It is applied to a modified Taylor vortex problem. Numerical examples confirm the theoretical results and show the accuracy of the method.

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1. Introduction

Burgers' and Navier-Stokes equations are major objects of interest in computational fluid dynamics (CFD). During the last five decades various numerical methods have been developed for simulating viscous incompressible flows governed by these nonlinear equations. Finite difference methods (FDM) turned out to be very popular, since they are easily implemented in various situations. In the case of Navier-Stokes equations, the majority of finite difference methods have the second order accuracy that is sufficient for most of CFD problems. Among the most popular are conventional second order central and upwind schemes. For problems with smooth well-behaved solutions, these methods deliver good

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results on uniform grids. On the other hand, in convection dominated problems, they behave poorly if the mesh is not sufficiently refined. In addition, non-compact higher order discretisations require 5-grid points in both x - and y -directions that cannot be guaranteed near boundary nodes. Therefore, it is important to have higher order compact schemes using 3-grid points adjacent to the boundary. Higher order compact methods are more efficient and provide more accurate numerical results.

For the time-dependent Navier-Stokes equations of motion a number of explicit and implicit methods are developed by Hirsh [9], Rai and Moin [25] and Lele [14]. These methods have fourth order accuracy in the spatial direction and second order accuracy in the time direction. For 2D advection dominated flows, Balzano [3] discussed an explicit compact method with second order time accuracy. Although explicit methods are easily implementable, they have a conditional stability limit in time step. Implicit schemes are unconditionally stable but they require matrix inversion at each advanced time level. For 1D and 2D time-dependent parabolic problems, several higher order implicit schemes are studied by Mohanty *et al.* [18,21,22], Strickwerda [30], Yanwen *et al.* [33] and Shah *et al.* [26]. Using stream-function-vorticity or stream-function-velocity formulation, Ghia *et al.* [8], Lecointe and Piquet [13], Li *et al.* [15], Spatz and Carey [29], Spatz [28], Weinan and Liu [32], Meitz and Fasel [16], Erturk and Gokcol [6], Mohanty *et al.* [17] solved the incompressible Navier-Stokes equations.

However, in 3D case such a formulation increases the number of equations and unknowns that results in higher computational cost. Tafti [31] developed an alternate formulation for the pressure equation in Laplacian form on a collocated grid for the solution of the incompressible Navier-Stokes equations. Johnson and Liu [11] studied a method for incompressible flow based on local pressure boundary conditions. A higher order finite volume method was employed by Pereira *et al.* [23], spectral method by Peyret [24], high order explicit upwind compact scheme and UGS solution algorithm by Bai *et al.* [2] in the artificial compressibility method. Other high order finite difference methods for the solution of incompressible fluid flows are discussed in [1, 4, 5, 7, 10, 27].

The aim of the present work is to solve 2D time-dependent viscous Burgers' and Navier-Stokes equations of motion with appropriate initial and Dirichlet boundary conditions by a high order compact method. We propose a new exponential implicit method for general 2D nonlinear parabolic equations in line with the 2D nonlinear schemes for elliptic equations — cf. [19, 20]. The method involves only two time levels and has accuracy of order two in time and order four in space. We construct an exponentially fitted method at each time level. At advanced time levels this method uses only nine grid points of a single compact cell with minimal stencil width in the x - and y -directions. Numerical simulations verify the usefulness of the proposed scheme in terms of maximum absolute (MA) errors.

The paper is arranged as follows. Section 2 deals with the discretisation of nonlinear 2D parabolic equations. The application and two-level nonlinear implicit schemes for the Burgers' and Navier-Stokes equations are discussed in Section 3. Stability of the method is considered in Section 4. Section 5 contains the results of numerical simulations. Finally, Section 6 provides the summary of this study.

2. Discretisation Procedure

Let v be a positive real number. We consider the following initial boundary value problem in the semi-infinite region $\Omega := \{(x, y, t) : 0 < x < 1, 0 < y < 1, t > 0\}$:

$$v \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = \frac{\partial \phi}{\partial t} + \psi \left(x, y, t, \phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right), \tag{2.1}$$

$$\begin{aligned} \phi(x, y, 0) &= f(x, y), & 0 \leq x, y \leq 1, \\ \phi(x, 0, t) &= a_0(x, t), \quad \phi(x, 1, t) = a_1(x, t), & 0 \leq x \leq 1, \quad t > 0, \\ \phi(0, y, t) &= b_0(y, t), \quad \phi(1, y, t) = b_1(y, t), & 0 \leq y \leq 1, \quad t > 0, \end{aligned} \tag{2.2}$$

where $\phi(x, y, t) \in C^6(\Omega)$, and $f(x, y), a_0(x, t), a_1(x, t), b_0(y, t), b_1(y, t)$ are smooth functions on the boundary of Ω .

Choosing $h > 0, \tau > 0$ and a positive integer M such that $(M + 1)h = 1$, we define a rectangular grid (x_i, y_j, t_n) , with the internal grid points $x_i = ih, y_j = jh, t_n = n\tau, i, j = 0, \dots, M + 1$ and $n = 0, 1, 2, \dots$. Besides, let λ denote the mesh ratio τ/h^2 and $\phi_{i,j}^n$ be an approximation of the exact value $\phi_{i,j}^n$ of the function $\phi(x, y, t)$ at the grid point (x_i, y_j, t_n) . The differential equation (2.1) can be now approximated as

$$v \left(\frac{\partial^2 \phi_{i,j}^n}{\partial x^2} + \frac{\partial^2 \phi_{i,j}^n}{\partial y^2} \right) - \frac{\partial \phi_{i,j}^n}{\partial t} = \psi \left(x_i, y_j, t_n, \phi_{i,j}^n, \frac{\partial \phi_{i,j}^n}{\partial x}, \frac{\partial \phi_{i,j}^n}{\partial y} \right) \equiv \Psi_{i,j}^n. \tag{2.3}$$

Let $\bar{t}_n = t_n + \tau/2$ and

$$\bar{\phi}_{i,j}^n = \frac{1}{2} (\phi_{i,j}^{n+1} + \phi_{i,j}^n), \quad \bar{\phi}_{i\pm 1,j}^n = \frac{1}{2} (\phi_{i\pm 1,j}^{n+1} + \phi_{i\pm 1,j}^n), \tag{2.4}$$

$$\bar{\phi}_{i,j\pm 1}^n = \frac{1}{2} (\phi_{i,j\pm 1}^{n+1} + \phi_{i,j\pm 1}^n), \quad \bar{\phi}_{t,i,j}^n = \frac{1}{\tau} (\phi_{i,j}^{n+1} - \phi_{i,j}^n), \tag{2.5}$$

$$\bar{\phi}_{t,i\pm 1,j}^n = \frac{1}{\tau} (\phi_{i\pm 1,j}^{n+1} - \phi_{i\pm 1,j}^n), \quad \bar{\phi}_{t,i,j\pm 1}^n = \frac{1}{\tau} (\phi_{i,j\pm 1}^{n+1} - \phi_{i,j\pm 1}^n), \tag{2.6}$$

$$\bar{\phi}_{x,i,j}^n = \frac{1}{2h} (\bar{\phi}_{i+1,j}^{n+1} - \bar{\phi}_{i-1,j}^n), \quad \bar{\phi}_{x,i\pm 1,j}^n = \frac{1}{2h} (\pm 3\bar{\phi}_{i\pm 1,j}^n \mp 4\bar{\phi}_{i,j}^n \pm \bar{\phi}_{i\mp 1,j}^n), \tag{2.7}$$

$$\bar{\phi}_{x,i,j\pm 1}^n = \frac{1}{2h} (\bar{\phi}_{i+1,j\pm 1}^{n+1} - \bar{\phi}_{i-1,j\pm 1}^n), \quad \bar{\phi}_{y,i,j}^n = \frac{1}{2h} (\bar{\phi}_{i,j+1}^{n+1} - \bar{\phi}_{i,j-1}^n), \tag{2.8}$$

$$\bar{\phi}_{y,i\pm 1,j}^n = \frac{1}{2h} (\bar{\phi}_{i\pm 1,j+1}^{n+1} - \bar{\phi}_{i\pm 1,j-1}^n), \quad \bar{\phi}_{y,i,j\pm 1}^n = \frac{1}{2h} (\pm 3\bar{\phi}_{i,j\pm 1}^n \mp 4\bar{\phi}_{i,j}^n \pm \bar{\phi}_{i,j\mp 1}^n), \tag{2.9}$$

$$\bar{\phi}_{xx,i,j}^n = \frac{1}{h^2} (\bar{\phi}_{i+1,j}^n - 2\bar{\phi}_{i,j}^n + \bar{\phi}_{i-1,j}^n), \quad \bar{\phi}_{xx,i,j\pm 1}^n = \frac{1}{h^2} (\bar{\phi}_{i+1,j\pm 1}^n - 2\bar{\phi}_{i,j\pm 1}^n + \bar{\phi}_{i-1,j\pm 1}^n), \tag{2.10}$$

$$\bar{\phi}_{yy,i,j}^n = \frac{1}{h^2} (\bar{\phi}_{i,j+1}^n - 2\bar{\phi}_{i,j}^n + \bar{\phi}_{i,j-1}^n), \quad \bar{\phi}_{yy,i\pm 1,j}^n = \frac{1}{h^2} (\bar{\phi}_{i\pm 1,j+1}^n - 2\bar{\phi}_{i\pm 1,j}^n + \bar{\phi}_{i\pm 1,j-1}^n). \tag{2.11}$$

Besides, let $\delta_x \phi_i = \phi_{i+1/2} - \phi_{i-1/2}$ and $\mu_x \phi_i = (1/2)(\phi_{i+1/2} + \phi_{i-1/2})$ be, respectively, central difference and averaging operators in the x -direction.

Applying the fourth-order compact scheme to the second derivatives in (2.3) and using algebraic manipulations of [19], we write the Eq. (2.3) as

$$\begin{aligned} & v \left(\delta_x^2 + \delta_y^2 + \frac{1}{6} \delta_x^2 \delta_y^2 \right) \phi_{i,j}^n \\ &= h^2 \phi_{t_{i,j}}^n \exp \left[\frac{\phi_{t_{i+1,j}}^n + \phi_{t_{i-1,j}}^n + \phi_{t_{i,j+1}}^n + \phi_{t_{i,j-1}}^n - 4\phi_{t_{i,j}}^n}{12\phi_{t_{i,j}}^n} \right] \\ &+ h^2 \psi_{i,j}^n \exp \left[\frac{\psi_{i+1,j}^n + \psi_{i-1,j}^n + \psi_{i,j+1}^n + \psi_{i,j-1}^n - 4\psi_{i,j}^n}{12\psi_{i,j}^n} \right] + \mathcal{O}(h^6). \end{aligned} \tag{2.12}$$

At the grid points (x_i, y_j, t_n) , we denote

$$\begin{aligned} \phi_{abc} &= \frac{\partial^{(a+b+c)} \phi_{i,j}^n}{\partial x^a \partial y^b \partial t^c}, \quad a, b, c = 0, 1, 2, \dots, \\ \alpha &= \psi_{t_{i,j}}^n, \quad \beta = \psi_{\phi_{i,j}}^n, \quad \gamma = \psi_{\phi_{x_{i,j}}}^n, \quad \eta = \psi_{\phi_{y_{i,j}}}^n. \end{aligned} \tag{2.13}$$

Differentiating the Eq. (2.1) with respect to t and using (2.13) yields

$$v(\phi_{201} + \phi_{021}) = \phi_{002} + \alpha + \beta \phi_{001} + \gamma \phi_{101} + \eta \phi_{011}. \tag{2.14}$$

Next, we define the approximations

$$\begin{aligned} \bar{\psi}_{i\pm 1,j}^n &= \psi(x_{i\pm 1}, y_j, t_n, \bar{\phi}_{i\pm 1,j}^n, \bar{\phi}_{x_{i\pm 1,j}}^n, \bar{\phi}_{y_{i\pm 1,j}}^n), \\ \bar{\psi}_{i,j\pm 1}^n &= \psi(x_i, y_{j\pm 1}, t_n, \bar{\phi}_{i,j\pm 1}^n, \bar{\phi}_{x_{i,j\pm 1}}^n, \bar{\phi}_{y_{i,j\pm 1}}^n). \end{aligned} \tag{2.15}$$

Using (2.4)-(2.11) and (2.13), we simplify (2.15), thus obtaining

$$\begin{aligned} \bar{\psi}_{i\pm 1,j}^n &= \psi_{i\pm 1,j}^n + \frac{\tau}{2}(\alpha + \beta \phi_{001} + \gamma \phi_{101} + \eta \phi_{011}) \\ &+ \frac{h^2}{6} T_1 + \mathcal{O}(\pm \tau h \pm h^3 + \tau^2 + \tau h^2 + h^4), \\ \bar{\psi}_{i,j\pm 1}^n &= \psi_{i,j\pm 1}^n + \frac{\tau}{2}(\alpha + \beta \phi_{001} + \gamma \phi_{101} + \eta \phi_{011}) \\ &+ \frac{h^2}{6} T_2 + \mathcal{O}(\pm \tau h \pm h^3 + \tau^2 + \tau h^2 + h^4), \end{aligned} \tag{2.16}$$

where

$$T_1 = -2\gamma \phi_{300} + \eta \phi_{030}, \quad T_2 = \gamma \phi_{300} - 2\eta \phi_{030}. \tag{2.17}$$

Next we define the terms

$$\begin{aligned} \bar{\phi}_{x_{i,j}}^n &= \bar{\phi}_{x_{i,j}}^n + a_1 h \left[(\bar{\phi}_{t_{i+1,j}}^n - \bar{\phi}_{t_{i-1,j}}^n) + (\bar{\psi}_{i+1,j}^n - \bar{\psi}_{i-1,j}^n) \right] \\ &+ a_2 h (\bar{\phi}_{y_{y_{i+1,j}}}^n - \bar{\phi}_{y_{y_{i-1,j}}}^n), \\ \bar{\phi}_{y_{i,j}}^n &= \bar{\phi}_{y_{i,j}}^n + b_1 h \left[(\bar{\phi}_{t_{i,j+1}}^n - \bar{\phi}_{t_{i,j-1}}^n) + (\bar{\psi}_{i,j+1}^n - \bar{\psi}_{i,j-1}^n) \right] \\ &+ b_2 h (\bar{\phi}_{xx_{i,j+1}}^n - \bar{\phi}_{xx_{i,j-1}}^n), \end{aligned} \tag{2.18}$$

where a_1, a_2, b_1 and b_2 are parameters to be determined.

Taking into account the Eqs. (2.4)-(2.11) and (2.16), we write (2.18) as

$$\begin{aligned} \bar{\phi}_{x_{i,j}}^n &= \phi_{x_{i,j}}^n + \frac{\tau}{2}\phi_{101} + \frac{h^2}{6}(1 + 12va_1)\phi_{300} \\ &\quad + 2h^2(va_1 + a_2)\phi_{120} + \mathcal{O}(\tau^2 + \tau h^2 + h^4), \\ \bar{\phi}_{y_{i,j}}^n &= \phi_{y_{i,j}}^n + \frac{\tau}{2}\phi_{011} + \frac{h^2}{6}(1 + 12vb_1)\phi_{030} \\ &\quad + 2h^2(vb_1 + b_2)\phi_{210} + \mathcal{O}(\tau^2 + \tau h^2 + h^4). \end{aligned}$$

For $a_1 = b_1 = -1/12v, a_2 = b_2 = 1/12$, the above equations take the form

$$\begin{aligned} \bar{\phi}_{x_{i,j}}^n &= \phi_{x_{i,j}}^n + \frac{\tau}{2}\phi_{101} + \mathcal{O}(\tau^2 + \tau h^2 + h^4), \\ \bar{\phi}_{y_{i,j}}^n &= \phi_{y_{i,j}}^n + \frac{\tau}{2}\phi_{011} + \mathcal{O}(\tau^2 + \tau h^2 + h^4). \end{aligned} \tag{2.19}$$

Next we define

$$\bar{\psi}_{i,j}^n = \psi(x_i, y_j, t_n, \bar{\phi}_{i,j}^n, \bar{\phi}_{x_{i,j}}^n, \bar{\phi}_{y_{i,j}}^n). \tag{2.20}$$

It follows from (2.4), (2.19), (2.20) that

$$\bar{\psi}_{i,j}^n = \psi_{i,j}^n + \frac{\tau}{2}(\alpha + \beta\phi_{001} + \gamma\phi_{101} + \eta\phi_{011}) + \mathcal{O}(\tau^2 + \tau h^2 + h^4). \tag{2.21}$$

Let

$$\begin{aligned} \hat{\phi}_{x_{i,j}}^n &= \bar{\phi}_{x_{i,j}}^n + a_3h\left[(\bar{\phi}_{t_{i+1,j}}^n - \bar{\phi}_{t_{i-1,j}}^n) + (\bar{\psi}_{i+1,j}^n - \bar{\psi}_{i-1,j}^n)\right] \\ &\quad + a_4h(\bar{\phi}_{yy_{i+1,j}}^n - \bar{\phi}_{yy_{i-1,j}}^n), \\ \hat{\phi}_{y_{i,j}}^n &= \bar{\phi}_{y_{i,j}}^n + b_3h\left[(\bar{\phi}_{t_{i,j+1}}^n - \bar{\phi}_{t_{i,j-1}}^n) + (\bar{\psi}_{i,j+1}^n - \bar{\psi}_{i,j-1}^n)\right] \\ &\quad + b_4h(\bar{\phi}_{xx_{i,j+1}}^n - \bar{\phi}_{xx_{i,j-1}}^n), \end{aligned} \tag{2.22}$$

where a_3, a_4, b_3 and b_4 are parameters to be determined. The Eqs. (2.22) can be written as

$$\begin{aligned} \hat{\phi}_{x_{i,j}}^n &= \phi_{x_{i,j}}^n + \frac{\tau}{2}\phi_{101} + \frac{h^2}{6}T_3 + \mathcal{O}(\tau^2 + \tau h^2 + h^4), \\ \hat{\phi}_{y_{i,j}}^n &= \phi_{y_{i,j}}^n + \frac{\tau}{2}\phi_{011} + \frac{h^2}{6}T_4 + \mathcal{O}(\tau^2 + \tau h^2 + h^4), \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} T_3 &= (1 + 12va_3)\phi_{300} + 12(va_3 + a_4)\phi_{120}, \\ T_4 &= (1 + 12vb_3)\phi_{030} + 12(vb_3 + b_4)\phi_{210}. \end{aligned} \tag{2.24}$$

Introducing the term

$$\hat{\psi}_{i,j}^n = \psi(x_i, y_j, t_n, \bar{\phi}_{i,j}^n, \hat{\phi}_{x_{i,j}}^n, \hat{\phi}_{y_{i,j}}^n) \tag{2.25}$$

and using the relations (2.4) and (2.23), we write it as

$$\begin{aligned} \hat{\psi}_{i,j}^n = & \psi_{i,j}^n + \frac{\tau}{2}(\alpha + \beta\phi_{001} + \gamma\phi_{101} + \eta\phi_{011}) \\ & + \frac{h^2}{6}(\gamma T_3 + \eta T_4) + \mathcal{O}(\tau^2 + \tau h^2 + h^4). \end{aligned} \tag{2.26}$$

Then at each internal node (x_i, y_j, t_n) , the differential equation (2.1) is discretised by

$$\begin{aligned} & v \left(\delta_x^2 + \delta_y^2 + \frac{1}{6} \delta_x^2 \delta_y^2 \right) \bar{\phi}_{i,j}^n \\ = & h^2 \bar{\phi}_{t_{i,j}}^n \exp \left[\frac{\bar{\phi}_{t_{i+1,j}}^n + \bar{\phi}_{t_{i-1,j}}^n + \bar{\phi}_{t_{i,j+1}}^n + \bar{\phi}_{t_{i,j-1}}^n - 4\bar{\phi}_{t_{i,j}}^n}{12\bar{\phi}_{t_{i,j}}^n} \right] \\ & + h^2 \bar{\psi}_{i,j}^n \exp \left[\frac{\bar{\psi}_{i+1,j}^n + \bar{\psi}_{i-1,j}^n + \bar{\psi}_{i,j+1}^n + \bar{\psi}_{i,j-1}^n - 4\hat{\psi}_{i,j}^n}{12\bar{\psi}_{i,j}^n} \right] + \bar{T}_{i,j}^n, \end{aligned} \tag{2.27}$$

where $\bar{T}_{i,j}^n = \mathcal{O}(\tau^2 h^2 + \tau h^4 + h^6)$ and

$$\delta_x^2 \bar{\phi}_{i,j}^n = \delta_x^2 \phi_{i,j}^n + \frac{\tau h^2}{2} \phi_{201} + \mathcal{O}(\tau^2 h^2 + \tau h^4), \tag{2.28}$$

$$\delta_y^2 \bar{\phi}_{i,j}^n = \delta_y^2 \phi_{i,j}^n + \frac{\tau h^2}{2} \phi_{021} + \mathcal{O}(\tau^2 h^2 + \tau h^4), \tag{2.29}$$

$$\delta_x^2 \delta_y^2 \bar{\phi}_{i,j}^n = \delta_x^2 \delta_y^2 \phi_{i,j}^n + \mathcal{O}(\tau h^4). \tag{2.30}$$

Taking into account the second formula in (2.5) and the Eqs. (2.6), (2.16), (2.21), (2.26), (2.28)-(2.30), we obtain from(2.12), (2.27) that

$$\begin{aligned} & \frac{v\tau h^2}{2}(\phi_{201} + \phi_{021}) + \mathcal{O}(\tau^2 h^2 + \tau h^4 + h^6) \\ = & \frac{h^2}{12} \left[6\tau(\phi_{002} + \alpha + \beta\phi_{001} + \gamma\phi_{101} + \eta\phi_{011}) + \frac{h^2}{3}(T_1 + T_2 - 2\gamma T_3 - 2\eta T_4) \right] + \bar{T}_{i,j}^n. \end{aligned} \tag{2.31}$$

Finally, the relations (2.17), (2.24) and (2.14) in (2.27) allow us to evaluate the error term $\bar{T}_{i,j}^n$ in (2.31) as

$$\begin{aligned} \bar{T}_{i,j}^n = & \frac{h^4}{12} \left[(1 + 8va_3)\gamma\phi_{300} + 8(va_3 + a_4)\gamma\phi_{120} \right. \\ & \left. + (1 + 8vb_3)\eta\phi_{030} + 8(vb_3 + b_4)\eta\phi_{210} \right] \\ & + \mathcal{O}(\tau^2 h^2 + \tau h^4 + h^6). \end{aligned} \tag{2.32}$$

The method (2.27) is of order $\mathcal{O}(\tau^2 + \tau h^2 + h^4)$, if the coefficient at h^4 in (2.32) vanishes. This leads to the system of equations

$$1 + 8va_3 = 0, \quad va_3 + a_4 = 0, \quad 1 + 8vb_3 = 0, \quad vb_3 + b_4 = 0,$$

the solution of which is

$$a_3 = b_3 = \frac{-1}{8\nu}, \quad a_4 = b_4 = \frac{1}{8},$$

so that the local truncation error is

$$\bar{T}_{i,j}^n = \mathcal{O}(\tau^2 h^2 + \tau h^4 + h^6).$$

Systems of nonlinear parabolic equations and equations of higher dimensions can be discretised analogously.

Incorporating the conditions (2.2) into the formula (2.27), we obtain a tri-block-diagonal system of equations. Note that linear differential equations produce linear systems — cf. [21,22], and we use the ADI method to solve it. For nonlinear differential equations one obtains non-linear systems — cf. [12], and we use the Newton iterative method to solve it. It is worth noting that for a fixed mesh ratio, the formula (2.27) has the fourth order of spatial accuracy.

3. Two-Level Schemes for Time-Dependent Burgers' and Navier-Stokes Equations

Burgers' equation is a fundamental time-dependent nonlinear parabolic PDE, which appears in various areas of applied mathematics, including traffic flow, heat conduction, fluid mechanics and nonlinear acoustic waves. It represents nonlinear physical problems and it is difficult to solve it exactly. In the last two decades, significant efforts have been spent on developing numerical methods for its solution. Here, we apply the scheme (2.27) to two-dimensional unsteady viscous Burgers' equation

$$R_e^{-1}(\phi_{xx} + \phi_{yy}) = \phi_t + \phi(\phi_x + \phi_y), \quad 0 < x, y < 1, \quad t > 0, \quad (3.1)$$

where R_e is Reynolds number and $\nu = R_e^{-1}$ the viscosity coefficient. Using (2.27) in (3.1) gives

$$\begin{aligned} & \nu \left(\delta_x^2 + \delta_y^2 + \frac{1}{6} \delta_x^2 \delta_y^2 \right) \bar{\phi}_{i,j}^n \\ &= \frac{h^2}{12} \left[\bar{\phi}_{t_{i+1,j}}^n + \bar{\phi}_{i+1,j}^n (\bar{\phi}_{x_{i+1,j}}^n + \bar{\phi}_{y_{i+1,j}}^n) + \bar{\phi}_{t_{i-1,j}}^n + \bar{\phi}_{i-1,j}^n (\bar{\phi}_{x_{i-1,j}}^n + \bar{\phi}_{y_{i-1,j}}^n) \right. \\ & \quad + \bar{\phi}_{t_{i,j+1}}^n + \bar{\phi}_{i,j+1}^n (\bar{\phi}_{x_{i,j+1}}^n + \bar{\phi}_{y_{i,j+1}}^n) + \bar{\phi}_{t_{i,j-1}}^n + \bar{\phi}_{i,j-1}^n (\bar{\phi}_{x_{i,j-1}}^n + \bar{\phi}_{y_{i,j-1}}^n) \\ & \quad \left. + 12\bar{\phi}_{t_{i,j}}^n + \bar{\phi}_{i,j}^n (\bar{\phi}_{x_{i,j}}^n + \bar{\phi}_{y_{i,j}}^n) - 4\bar{\phi}_{t_{i,j}}^n + \bar{\phi}_{i,j}^n (\hat{\phi}_{x_{i,j}}^n + \hat{\phi}_{y_{i,j}}^n) \right], \quad (3.2) \end{aligned}$$

where the subsidiary approximations of the Eq. (3.2) are discussed in Section 2. The Navier-Stokes equations play an importance role in fluid motion, so that various numerical techniques for fluid flows based on these equations have been developed in last decades. Numerical methods for 2D flows are very popular and many specialised schemes have been

developed. We consider two-dimensional Navier-Stokes equations of motion

$$v(u_{xx} + u_{yy}) = u_t + uu_x + vu_y + p_x, \tag{3.3}$$

$$v(v_{xx} + v_{yy}) = v_t + uv_x + vv_y + p_y, \tag{3.4}$$

where u and v are the unknown velocity components and $p(x, y, t)$ is the fluid pressure. Assuming that the pressure function is known and applying the method (2.27) in coupled form to the system of equations (3.3)-(3.4), we arrive at the following numerical scheme

$$\begin{aligned} & v \left(\delta_x^2 + \delta_y^2 + \frac{1}{6} \delta_x^2 \delta_y^2 \right) \bar{u}_{i,j}^n \\ &= \frac{h^2}{12} \left[\bar{u}_{t_{i+1,j}}^n + \bar{u}_{i+1,j}^n \bar{u}_{x_{i+1,j}}^n + \bar{v}_{i+1,j}^n \bar{u}_{y_{i+1,j}}^n + \bar{p}_{x_{i+1,j}}^n + \bar{u}_{t_{i-1,j}}^n + \bar{u}_{i-1,j}^n \bar{u}_{x_{i-1,j}}^n \right. \\ & \quad + \bar{v}_{i-1,j}^n \bar{u}_{y_{i-1,j}}^n + \bar{p}_{x_{i-1,j}}^n + \bar{u}_{t_{i,j+1}}^n + \bar{u}_{i,j+1}^n \bar{u}_{x_{i,j+1}}^n + \bar{v}_{i,j+1}^n \bar{u}_{y_{i,j+1}}^n + \bar{p}_{x_{i,j+1}}^n + \bar{u}_{t_{i,j-1}}^n \\ & \quad + \bar{u}_{i,j-1}^n \bar{u}_{x_{i,j-1}}^n + \bar{v}_{i,j-1}^n \bar{u}_{y_{i,j-1}}^n + \bar{p}_{x_{i,j-1}}^n + 12 \left(\bar{u}_{t_{i,j}}^n + \bar{u}_{i,j}^n \bar{u}_{x_{i,j}}^n + \bar{v}_{i,j}^n \bar{u}_{y_{i,j}}^n + \bar{p}_{x_{i,j}}^n \right) \\ & \quad \left. - 4 \left(\bar{u}_{t_{i,j}}^n + \bar{u}_{i,j}^n \hat{u}_{x_{i,j}}^n + \bar{v}_{i,j}^n \hat{u}_{y_{i,j}}^n + \bar{p}_{x_{i,j}}^n \right) \right], \tag{3.5} \end{aligned}$$

$$\begin{aligned} & v \left(\delta_x^2 + \delta_y^2 + \frac{1}{6} \delta_x^2 \delta_y^2 \right) \bar{v}_{i,j}^n \\ &= \frac{h^2}{12} \left[\bar{v}_{t_{i+1,j}}^n + \bar{u}_{i+1,j}^n \bar{v}_{x_{i+1,j}}^n + \bar{v}_{i+1,j}^n \bar{v}_{y_{i+1,j}}^n + \bar{p}_{y_{i+1,j}}^n + \bar{v}_{t_{i-1,j}}^n + \bar{u}_{i-1,j}^n \bar{v}_{x_{i-1,j}}^n \right. \\ & \quad + \bar{v}_{i-1,j}^n \bar{v}_{y_{i-1,j}}^n + \bar{p}_{y_{i-1,j}}^n + \bar{v}_{t_{i,j+1}}^n + \bar{u}_{i,j+1}^n \bar{v}_{x_{i,j+1}}^n + \bar{v}_{i,j+1}^n \bar{v}_{y_{i,j+1}}^n + \bar{p}_{y_{i,j+1}}^n + \bar{v}_{t_{i,j-1}}^n \\ & \quad + \bar{u}_{i,j-1}^n \bar{v}_{x_{i,j-1}}^n + \bar{v}_{i,j-1}^n \bar{v}_{y_{i,j-1}}^n + \bar{p}_{y_{i,j-1}}^n + 12 \left(\bar{v}_{t_{i,j}}^n + \bar{u}_{i,j}^n \bar{v}_{x_{i,j}}^n + \bar{v}_{i,j}^n \bar{v}_{y_{i,j}}^n + \bar{p}_{y_{i,j}}^n \right) \\ & \quad \left. - 4 \left(\bar{v}_{t_{i,j}}^n + \bar{u}_{i,j}^n \hat{v}_{x_{i,j}}^n + \bar{v}_{i,j}^n \hat{v}_{y_{i,j}}^n + \bar{p}_{y_{i,j}}^n \right) \right], \tag{3.6} \end{aligned}$$

where the approximations of u and v associated with (3.5) and (3.6) are defined in Section 2 in terms of ϕ .

4. Stability Analysis

In order to study the stability, we use the linearised form of the Burgers' equation, i.e.

$$v(\phi_{xx} + \phi_{yy}) = \phi_t + \beta(\phi_x + \phi_y) \tag{4.1}$$

with a constant convective velocity $\beta > 0$. Introducing the cell Reynolds number $C_R = (\beta h)/(2\nu)$ and applying the method (2.27) to the differential equation (4.1), we obtain

$$\begin{aligned} & \left[1 + \frac{1}{12} (1 - 6\nu\lambda - 2\nu\lambda C_R^2) \delta_x^2 + \frac{1}{12} (1 - 6\nu\lambda - 2\nu\lambda C_R^2) \delta_y^2 \right. \\ & \quad \left. - \frac{C_R}{12} (1 - 6\nu\lambda) (2\mu_x \delta_x + 2\mu_y \delta_y) \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{v\lambda}{12} \left(2C_R \delta_y^2 \mu_x \delta_x + 2C_R \delta_x^2 \mu_y \delta_y - 4C_R^2 \mu_x \delta_x \mu_y \delta_y - \delta_x^2 \delta_y^2 \right) \Big] \phi_{i,j}^{n+1} \\
 = & \left[1 + \frac{1}{12} (1 + 6v\lambda + 2v\lambda C_R^2) \delta_x^2 + \frac{1}{12} (1 + 6v\lambda + 2v\lambda C_R^2) \delta_y^2 \right. \\
 & - \frac{C_R}{12} (1 + 6v\lambda) (2\mu_x \delta_x + 2\mu_y \delta_y) \\
 & \left. - \frac{v\lambda}{12} \left(2C_R \delta_y^2 \mu_x \delta_x + 2C_R \delta_x^2 \mu_y \delta_y - 4C_R^2 \mu_x \delta_x \mu_y \delta_y - \delta_x^2 \delta_y^2 \right) \right] \phi_{i,j}^n, \tag{4.2}
 \end{aligned}$$

where

$$\mu_x \phi_i = \frac{1}{2} (\phi_{i+(1/2)} + \phi_{i-(1/2)}), \quad \delta_x \phi_i = (\phi_{i+(1/2)} - \phi_{i-(1/2)}), \dots$$

This is an implicit scheme involving 9-grid points at the advanced time level $(n + 1)$. It can be written in the form

$$[X_1][Y_1]\phi_{i,j}^{n+1} = [X_2][Y_2]\phi_{i,j}^n, \tag{4.3}$$

where

$$\begin{aligned}
 X_1 &= \left[1 + \frac{1}{12} (1 - 6v\lambda - 2v\lambda C_R^2) \delta_x^2 - \frac{C_R}{12} (1 - 6v\lambda) (2\mu_x \delta_x) \right], \\
 Y_1 &= \left[1 + \frac{1}{12} (1 - 6v\lambda - 2v\lambda C_R^2) \delta_y^2 - \frac{C_R}{12} (1 - 6v\lambda) (2\mu_y \delta_y) \right], \\
 X_2 &= \left[1 + \frac{1}{12} (1 + 6v\lambda + 2v\lambda C_R^2) \delta_x^2 - \frac{C_R}{12} (1 + 6v\lambda) (2\mu_x \delta_x) \right], \\
 Y_2 &= \left[1 + \frac{1}{12} (1 + 6v\lambda + 2v\lambda C_R^2) \delta_y^2 - \frac{C_R}{12} (1 + 6v\lambda) (2\mu_y \delta_y) \right].
 \end{aligned}$$

There are additional higher orders terms in the Eq. (4.3), which do not affect the order of accuracy but allow to write (4.2) in factorization form of $\mathcal{O}(\tau^2 + h^4)$.

The scheme (4.3) in alternating direction implicit (ADI) form [21] may be written as

$$[X_1]\phi_{i,j}^* = [X_2][Y_2]\phi_{i,j}^n, \tag{4.4}$$

$$[Y_1]\phi_{i,j}^{n+1} = \phi_{i,j}^*, \tag{4.5}$$

where $\phi_{i,j}^*$ in (4.4), (4.5) is any dummy variable and the boundary conditions for finding $\phi_{i,j}^*$ can be obtained from (4.5).

To study the stability of (4.3), we employ the Fourier analysis. At each grid point, the error can be written in the form $\varepsilon_{(i,j)}^n = \xi^n \exp[i(i\theta_1 + j\theta_2)]$, where θ_1 and θ_2 are real angles and $i = \sqrt{-1}$. The characteristic equation for the Eq. (4.3) is

$$\xi = \xi_x \cdot \xi_y, \tag{4.6}$$

where

$$\begin{aligned}
 \xi_x &= \frac{1 - (1/3)(1 + 6v\lambda + 2v\lambda C_R^2) \sin^2(\theta_1/2) - (iC_R/6)(1 + 6v\lambda) \sin \theta_1}{1 - (1/3)(1 - 6v\lambda - 2v\lambda C_R^2) \sin^2(\theta_1/2) - (iC_R/6)(1 - 6v\lambda) \sin \theta_1}, \\
 \xi_y &= \frac{1 - (1/3)(1 + 6v\lambda + 2v\lambda C_R^2) \sin^2(\theta_2/2) - (iC_R/6)(1 + 6v\lambda) \sin \theta_2}{1 - (1/3)(1 - 6v\lambda - 2v\lambda C_R^2) \sin^2(\theta_2/2) - (iC_R/6)(1 - 6v\lambda) \sin \theta_2}.
 \end{aligned}$$

The equation is stable if $\xi^2 \leq 1$. It is shown in [4] that $|\xi_x|^2 \leq 1$ and the proof of $|\xi_y|^2 \leq 1$ is similar, so that (4.6) yields $|\xi|^2 \leq 1$. Thus the method above is unconditionally stable.

5. Numerical Tests

We now show the effectiveness of the scheme by applying it to four nonlinear engineering problems. Exact solutions of these problems are known and they help to establish initial and boundary values and evaluate the errors. Newton's block iteration method [12] is used to solve the system of non-linear difference equations. For all problems the initial guess is assumed trivial — i.e. $u = 0$. All computations are performed by using MATLAB code.

The error of the method is defined by

$$\|u_i^n - U_i^n\|_\infty = \max |u_i^n - U_i^n|,$$

where u_i^n and U_i^n are, respectively, numerical and exact solutions at the point (x_i, t_n) .

Example 5.1 (Viscous unsteady Burgers' equation).

$$v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t} + u \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right), \quad 0 < x < 1, \quad 0 < y < 1, \quad t > 0. \quad (5.1)$$

The exact solution is given by

$$u(x, y, t) = \frac{(2v\pi \sin[\pi(x+y)] \exp(-2v\pi^2 t))}{(2 + \cos[\pi(x+y)] \exp(-2v\pi^2 t))}.$$

For different v and $\lambda = 1.6$, Table 1 shows the maximum absolute errors for the proposed method and the method in [21]. Figs. 1 and 2 present numerical and exact solutions of the Eq. (5.1). Both plots are for the step size $h = 1/64$ and Reynolds number $R_e = v^{-1} = 10^6$.

Example 5.2 (Time-dependent Navier-Stokes equations of motion).

$$v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x}, \quad 0 < x, \quad y < \pi, \quad t > 0, \quad (5.2)$$

$$v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y}, \quad 0 < x, \quad y < \pi, \quad t > 0 \quad (5.3)$$

with the pressure function

$$p(x, y, t) = -\frac{1}{4}(\cos 2x + \cos 2y)e^{(-4vt)}.$$

Exact solutions is

$$u = -\cos x \sin y e^{(-2vt)}, \quad v = \sin x \cos y e^{(-2vt)}.$$

For various v and $\lambda = 1.6/\pi^2$, Table 2 shows the maximum absolute errors for the proposed method and the method in [21]. Figs. 3-6 demonstrate numerical and exact solutions.

Table 1: Example 5.1. Maximum absolute errors, $t = 1$.

h	τ	Proposed Method (2.27)			Method in [21]		
		$R_e = 10^2$	$R_e = 10^4$	$R_e = 10^6$	$R_e = 10^2$	$R_e = 10^4$	$R_e = 10^6$
1/16	1/160	7.1178e-07	3.6749e-10	3.7834e-14	1.4122e-06	4.3534e-09	8.9751e-12
1/32	1/640	4.7311e-08	2.6503e-11	2.8935e-15	8.5579e-08	2.5732e-10	5.9022e-13
1/62	1/2560	2.9963e-09	1.6737e-12	1.8508e-16	5.6412e-09	1.6554e-11	3.7842e-14
1/128	1/10240	1.8829e-10	1.0393e-13	1.1592e-17	3.8452e-10	1.1129e-12	2.7041e-15

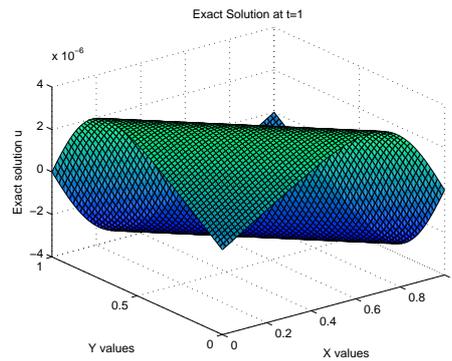
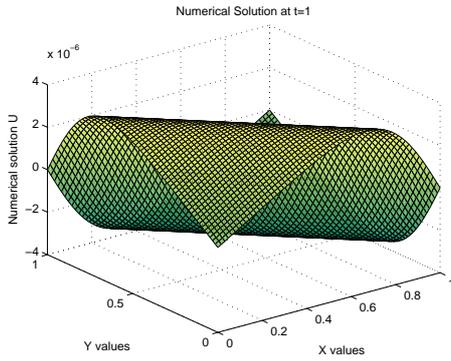


Figure 1: Example 5.1. Numerical solution, $h = 1/64$, $Re = 10^6$. Figure 2: Example 5.1. Exact solution, $h = 1/64$, $Re = 10^6$.

Table 2: Example 5.2. Maximum absolute errors, $t = 1$.

h	τ		Proposed Method (2.27)		Method in [21]	
			$v = 0.1$	$v = 0.02$	$v = 0.1$	$v = 0.02$
$\pi/16$	1/160	u	3.4768e-05	3.5985e-04	3.6922e-04	4.5524e-03
		v	5.8554e-05	7.7496e-04	4.8429e-04	5.3412e-03
$\pi/32$	1/640	u	2.1214e-06	2.2505e-05	2.3207e-05	2.7376e-04
		v	3.4888e-06	4.9773e-05	3.0288e-05	3.3120e-04
$\pi/64$	1/2560	u	1.3121e-07	1.4025e-06	1.4516e-06	1.7001e-05
		v	2.1532e-07	3.0864e-06	1.8932e-06	2.0667e-05
$\pi/128$	1/10240	u	8.1867e-09	8.7572e-08	9.0751e-08	1.0611e-06
		v	1.3426e-08	1.9238e-07	1.1886e-07	1.4114e-06

Example 5.3 (Taylor-Vortex Problem). We determine the approximate solution of the system (5.2)-(5.3) with the pressure function

$$p(x, y, t) = \frac{1}{4} [\cos(2Nx) + \cos(2Ny)] e^{(-4vN^2t)},$$

where N is the number of vortices. The exact solution is

$$u = -\cos(Nx) \sin(Ny) e^{(-2vN^2t)}, \quad v = \sin(Nx) \cos(Ny) e^{(-2vN^2t)}.$$

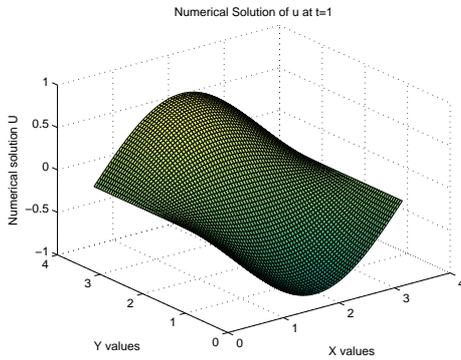


Figure 3: Example 5.2. Numerical solution for u ; $h = \pi/64, v = 0.1$.

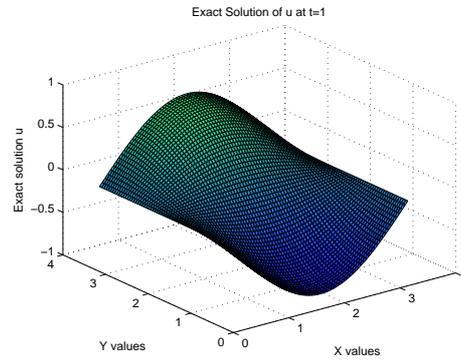


Figure 4: Example 5.2. Exact solution u ; $h = \pi/64, v = 0.1$.

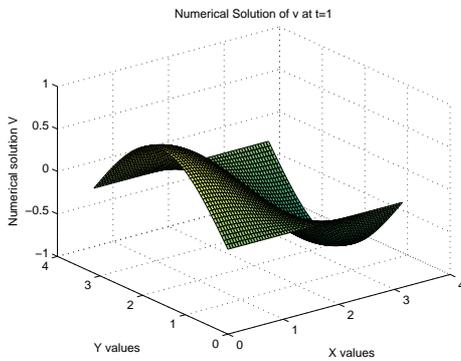


Figure 5: Example 5.2. Numerical solution for v ; $h = \pi/64, v = 0.1$.

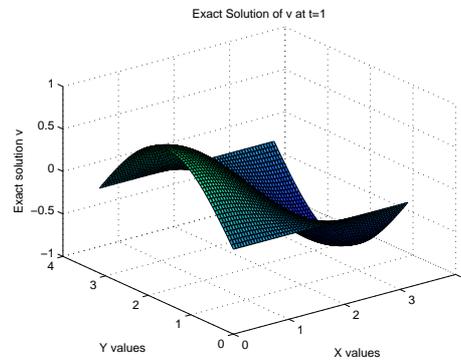


Figure 6: Example 5.2. Exact solution v ; $h = \pi/64, v = 0.1$.

Table 3: Example 5.3. Maximum absolute errors, $t = 1, N = 4$.

h	τ		Proposed Method (2.27)		Method in [21]	
			$v = 0.1$	$v = 0.02$	$v = 0.1$	$v = 0.02$
$\pi/16$	1/160	u	1.2738e-04	1.8535e-02	7.3752e-04	4.8356e-02
		v	1.2738e-04	1.8535e-02	7.3752e-04	4.8356e-02
$\pi/32$	1/640	u	9.9738e-06	8.9660e-04	4.6415e-05	2.9143e-03
		v	9.9738e-06	8.9660e-04	4.6415e-05	2.9143e-03
$\pi/64$	1/2560	u	6.5082e-07	3.8165e-05	2.9030e-06	1.8073e-04
		v	6.5082e-07	3.8165e-05	2.9030e-06	1.8073e-04
$\pi/128$	1/10240	u	4.1043e-08	2.3857e-06	1.8146e-07	1.1244e-05
		v	4.1043e-08	2.3857e-06	1.8146e-07	1.1244e-05

For various $v, \lambda = 1.6/\pi^2$ and $N = 4$, Table 3 shows the maximum absolute errors for the proposed method and the method in [21]. Figs. 7-10 display exact and approximate solutions.

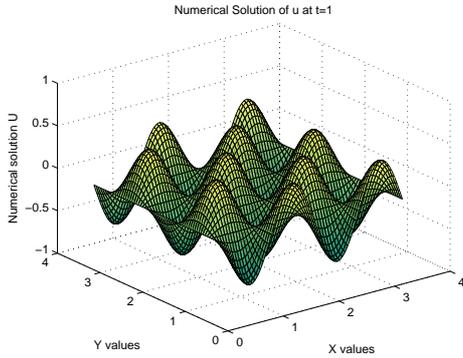


Figure 7: Example 5.3. Numerical solution for u ; $h = \pi/64$, $\nu = 0.02$, $N = 4$.

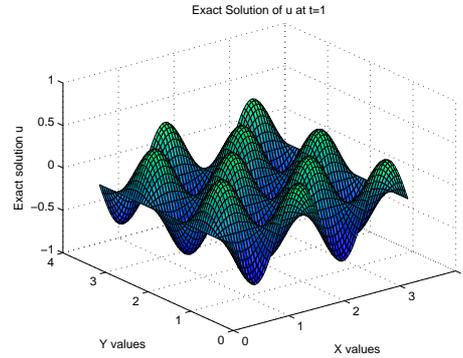


Figure 8: Example 5.3. Exact solution u ; $h = \pi/64$, $\nu = 0.02$ and $N = 4$.

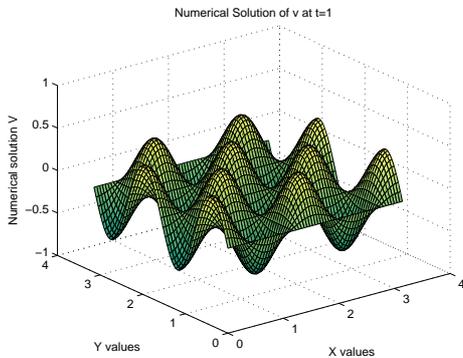


Figure 9: Example 5.3. Numerical solution for v ; $h = \pi/64$, $\nu = 0.02$, $N = 4$.

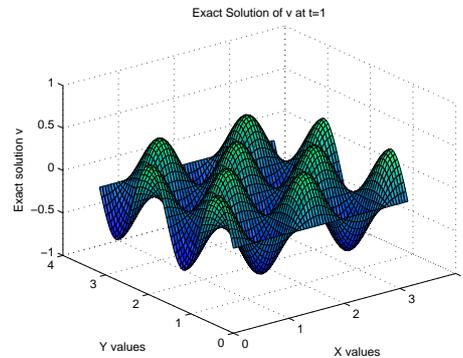


Figure 10: Example 5.3. Exact solution v ; $h = \pi/64$, $\nu = 0.02$, $N = 4$.

Example 5.4 (Unsteady Burgers' equation in cylindrical polar coordinates).

$$\frac{1}{R_e} \left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{r^2} u \right) = \frac{\partial u}{\partial t} + u \left(\frac{\partial u}{\partial r} + \frac{\partial u}{\partial z} \right) + f(r, z, t), \tag{5.4}$$

$$0 < r < 1, \quad 0 < z < 1, \quad t > 0.$$

The exact solution of this problem is

$$u(r, z, t) = \exp\left(\frac{-2\pi^2 t}{R_e}\right) \pi^2 r^2 \sin(\pi z).$$

We solve the Eq. (5.4) by methods (2.27) and [21]. For various R_e and $\lambda = 1.6$, Table 4 shows the maximum absolute errors for the proposed method and the method in [21]. Figs. 11 and 12 provide numerical and exact solutions of the Eq. (5.4). The spatial convergence order is computed as

$$\rho_h = \frac{\log(E_{h_1}/E_{h_2})}{\log(h_1/h_2)},$$

Table 4: Example 5.4. Maximum absolute errors, $t = 5$.

h	τ	Proposed Method (2.27)			Method in [21]		
		$R_e = 10$	$R_e = 50$	$R_e = 100$	$R_e = 10$	$R_e = 50$	$R_e = 100$
1/16	1/160	2.2839e-07	4.2703e-04	1.8115e-03	1.4892e-06	3.6713e-03	Over flow
1/32	1/640	1.4212e-08	2.6822e-05	1.1275e-04	9.3082e-08	2.2997e-04	Over flow
1/64	1/2560	8.8612e-10	1.6825e-06	7.0125e-06	5.8206e-09	1.4376e-05	Over flow
1/128	1/10240	5.5623e-11	1.0556e-07	4.4043e-07	3.6489e-10	8.9855e-07	Over flow

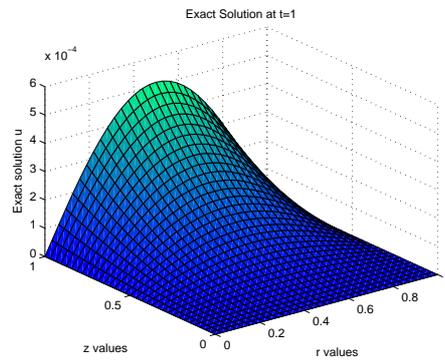
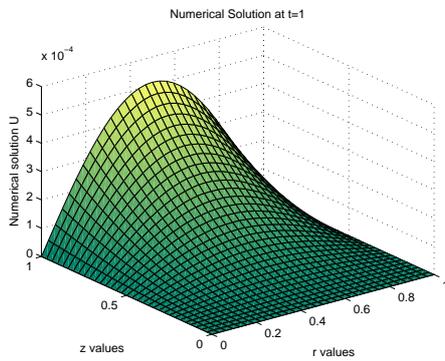


Figure 11: Example 5.4. Numerical solution for u ; Figure 12: Example 5.4. Exact solution u , $h = 1/32$, $R_e = 100$.

Table 5: Rate of convergence ρ_h in space.

Problem	Parameters, if any		Order of convergence
1 $h_1 = 1/64, h_2 = 1/128$	$R_e = 10^2$	for u	3.99
	$R_e = 10^4$	for u	4.01
	$R_e = 10^6$	for u	4.00
2 $h_1 = \pi/64, h_2 = \pi/128$	$v = 0.1$	for u	4.00
		for v	4.00
	$v = 0.02$	for u	4.00
		for v	4.00
3 $h_1 = \pi/64, h_2 = \pi/128$	$v = 0.1$	for u	3.99
		for v	3.99
	$v = 0.02$	for u	3.99
		for v	3.99
4 $h_1 = 1/64, h_2 = 1/128$	$R_e = 10$	for u	3.99
	$R_e = 50$	for u	3.99
	$R_e = 100$	for u	3.99

where E_{h_1} and E_{h_2} are maximum absolute errors for spatial uniform mesh sizes h_1 and h_2 , respectively.

The time convergence order is computed as

$$\rho_\tau = \frac{\log(E_{\tau_1}/E_{\tau_2})}{\log(\tau_1/\tau_2)},$$

where E_{τ_1} and E_{τ_2} are maximum absolute errors for time uniform mesh of size τ_1 and τ_2 , respectively.

Different values of ρ_h and ρ_τ for all the problems are presented in Tables 5 and 6, respectively. The log-log error plots for all problems are given in Figs. 13-16.

Table 6: Rate of convergence ρ_τ in time.

Problem	Parameters, if any		Order of convergence
1 $\tau_1 = 1/2560, \tau_2 = 1/10240$	$R_e = 10^2$	for u	1.996
	$R_e = 10^4$	for u	2.004
	$R_e = 10^6$	for u	1.998
2 $\tau_1 = 1/2560, \tau_2 = 1/10240$	$v = 0.1$	for u	2.001
		for v	2.001
	$v = 0.02$	for u	2.000
		for v	2.001
3 $\tau_1 = 1/2560, \tau_2 = 1/10240$	$v = 0.1$	for u	1.993
		for v	1.993
	$v = 0.02$	for u	1.999
		for v	1.999
4 $\tau_1 = 1/2560, \tau_2 = 1/10240$	$R_e = 10$	for u	1.996
	$R_e = 50$	for u	1.997
	$R_e = 100$	for u	1.996

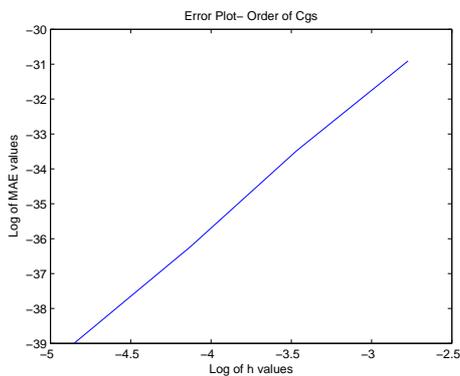


Figure 13: Example (5.1). log-log error.

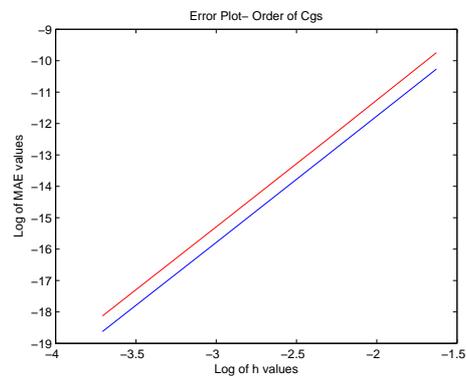


Figure 14: Example (5.2). log-log error.

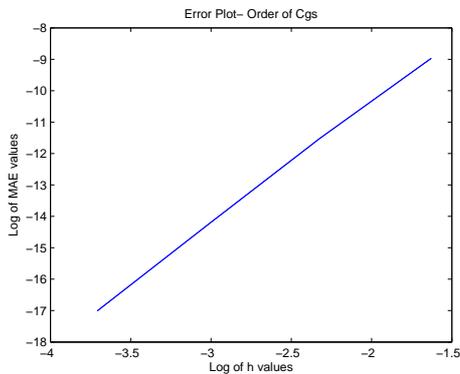


Figure 15: Example (5.3). log-log error.

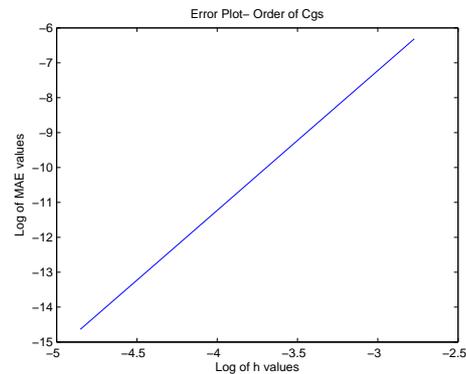


Figure 16: Example (5.4). log-log error.

6. Summary

We use an implicit exponential numerical scheme for solving unsteady 2D Burgers' equation, Navier-Stokes equations of motion and the Taylor-vortex problem. The scheme has fourth order accuracy in space and second order accuracy in time. It is compact and at the advanced time levels computational stencil requires only nine points. It is shown that for linearised Burgers' equation the method is unconditionally stable. Numerical simulations show that the scheme can produce oscillation-free solution for high Reynolds numbers — cf. [21, 22]. The simulations are in excellent match with analytic results. The method can be extended to more complex flow problems in polar coordinates and to complete Navier-Stokes equations with the pressure as a variable.

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