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CONVERGENCE ANALYSIS OF PSEUDOSPECTRAL METHOD WITH RESTRAINT OPERATOR FOR FLUID FLOW WITH LOW MACH NUMBER

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Abstract. In this paper, we propose a three level Pseudospectral scheme with restraint operator to solve the periodic problem of fluid flow with low Mach number. The generalized stability of the scheme is analyzed and the convergence is proved. Numerical results are presented.

Introduction

The fluid flow with low Mach number is governed by the following partial differential equations:

$$\begin{cases} \frac{\partial U}{\partial t}(x,t) + (U(x,t)\cdot\nabla)U(x,t) + \nabla P(x,t) - \nu\nabla^2 U(x,t) = f(x,t), \\ \frac{\partial P}{\partial t}(x,t) + (U(x,t)\cdot\nabla)P(x,t) = 0, \quad (x,t) \in \Omega \times (0,T], \\ U(x,0) = U_0(x), \qquad P(x,0) = P_0(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where $\Omega = (0, 2\pi)^2$, $\nu > 0$, is the viscosity coefficient. The speed vector and the pressure are denoted by $U = (U_1, U_2)$ and P respectively. The functions U_0, P_0 and f are given with period 2π for all the space variables.

Spectral methods are classical and largely used technique to solve differential equations, both theoretically and numerically. In recent years, spectral and pseudospectral methods have become very popular with their applications to computational fluid dynamics [1-4,7]. The pseudospectral methods are easier to implement for nonlinear partial differential equations. But they are not stable as the spectral ones due to "aliasing" especially for the flows with low Mach number. Therefore some authors proposed the filtering technique [6,9] to remedy the deficiency of instability.

The aim of this paper is to consider the periodic boundary value problem of fluid flow with low Mach number. A three-level pseudospectral scheme with restraint operator in combination with second order time differencing technique is constructed for fluid flow with low Mach number. The rate of convergence of the resulting scheme is $O(\tau^2 + N^{-s})$, where s depends only on the smoothness of the exact solution.

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1. The Scheme

Let

$$(u,v)=\frac{1}{(4\pi)^2}\int_\Omega u(x)\overline{v(x)}dx$$

$$\|u\|^2=(u,v)\qquad |u|_1=\left\|\frac{\partial u}{\partial x}\right\|.$$

Let Z be the set of integers. For $k=(k_1,k_2)\in Z^2$, $\quad k{\cdot}x=k_1{\cdot}x_1+k_2{\cdot}x_2$,

$$|k|_\infty=\max(|k_1|+|k_2|),\qquad |k|=(k_1+k_2)^{1/2}.$$

For any positive integer N, we define

$$V_N=\mathrm{span}\{e^{\iota k\cdot x}|k{\in}Z^2,|k|_\infty{\le}N\},$$
$$W_N=\mathrm{span}\{e^{\iota k\cdot x}|k{\in}Z^2,|k|{\le}N\},$$

let $h=\frac{2\pi}{2N+1}$ be the mesh size in the variable x and

$$\Omega_N=\{(ih,jh):i,j=0,1,2,...,2N\},$$

The discrete inner product and norm are defined respectively by

$$(u,v)_N=\frac{h}{2N+1}\sum_{j=0}^{2N}u(x_j)v(x_j),\quad \|\,u\,\|_N=(u,v)_N^{1/2},$$

$P_N:L^2(\overline{\Omega}){\mapsto}V_N$ be the orthogonal projection operator, i.e.,

$$(P_Nu,v)=(u,v),\qquad \forall v{\in}V_N,$$

$P_c:C(\Omega){\mapsto}V_N$ be the interpolation operator,i.e.,

$$P_cu(x_j)=u(x_j),\qquad \forall x_j{\in}\Omega_N.$$

Now, we define the restraint operator R_α, for $\alpha>1$, that is, if

$$u(x)=\sum_{|k|\le N}u_k\exp(\iota k{\cdot}x).$$

Then

$$R_\alpha u(x)=\sum_{|k|\le N}\left(1-\left[\left|\frac{k}{N}\right|\right]^\alpha\right)u_k\exp(2\pi\iota k{\cdot}x),$$

Let τ be the mesh size of the variable t and

$$R_t=\left\{t|t=k\tau,1{\le}k{\le}\left[\frac{T}{\tau}\right]\right\},$$

we denote u(x, t) by u(t) or u sometime. Let

$$u_{\hat{t}}=\frac{1}{2\tau}[u(t+\tau)-u(t-\tau)],\tag{2}$$

$$\widehat{u}(t)=\frac{1}{2}[u(t+\tau)+u(t-\tau)],\tag{3}$$

let u and p be the approximation to U and P respectively. To approach the non linear term $(U\cdot\bigtriangledown)U$ suitably, we define

$$J\left(u,v\right)=\frac{\partial}{\partial x_1}P_c\left(v^{(1)}u\right)+\frac{\partial}{\partial x_2}P_c\left(v^{(2)}u\right).\tag{4}$$

where $v^{(1)}, v^{(2)}$ are the components of v.
The pseudospectral scheme for solving (1) is to find $u(t) \in V_N$ and $p(t) \in W_N$, for $t \in R_\tau$, such that

$$\begin{cases} u_{\hat{t}}(t) + R_\alpha J\left(R_\alpha u(t), u(t)\right) + \nabla \widehat{p}(t) - \nu \nabla^2 \widehat{u}(t) = R_\alpha P_c f(t), \\ p_{\hat{t}}(t) + R_\alpha J\left(R_\alpha p(t), u(t)\right) = 0, \\ u(0) = P_c U_0, \qquad u(\tau) = P_c\left(U_0 + \tau \frac{\partial U}{\partial t}(0)\right), \\ p(0) = P_c P_0, \qquad p(\tau) = P_c\left(P_0 + \tau \frac{\partial P}{\partial t}(0)\right), \end{cases} \tag{5}$$

2. Main Theoretical Results

For error estimations, we need some notations, let

$$L^p(\Omega) = \left\{ u| \quad \| u \|_{L^p} = \left(\int_\Omega |u(x)|^p dx \right)^{1/p} < \infty \right\},$$

In particular, the inner product and norm are denoted by $(\cdot, \cdot)$ and $\| \cdot \|$ respectively for p = 2. For any positive integer, let $| u |_\mu = \left\| \frac{\partial^\mu u}{\partial x^\mu} \right\|$,

$$H^\mu(\Omega) = \left\{ u| \; \|u\| = \sum_{k=1}^{\mu} |u|_k < \infty \right\},$$

For any positive constant μ, $H^\mu(\Omega)$ is the complex interpolation space between $H^{[\mu]}(\Omega)$ and $H^{[\mu+1]}(\Omega)$. Define

$$H_p^\mu(\Omega) = \left\{ u| \, u \in H^\mu(\Omega), u(x) = u(x + 2\pi) \right\},$$

Let $C_p^\infty(\Omega)$ be the set of all infinity differentiable functions with period 2π for x_1, x_2. Clearly $H_p^\mu(\Omega)$ be the closure of $C_p^\infty(\Omega)$ in $H^\mu(\Omega)$.Let B be a Banach space. Define

$$L^2(0, T; B) = \left\{ u|u : [0, T] \to B, u \,\text{is strongly measurable and } \|u\|_{L^2(0,T;B)} < \infty \right\},$$

$$C(0, T; B) = \left\{ u|u : [0, T] \to B, u \text{ is strongly measurable and } \|u\|_{C(0,T;B)} < \infty \right\},$$

where

$$\|u\|_{L^2(0,T;B)} = \left(\int_0^T \|u\|_B^2 \, dt \right)^{1/2}, \quad \|u\|_{C(0,T;B)} = \max_{0 \le t \le T} \|u(t)\|_B$$

Moreover for any integer $S \ge 0$, let

$$H^S(0, T; B) = \left\{ u(x) \in L^2(0, T; B), \|u\|_{H^S(0,T;B)} < \infty \right\},$$

equipped with the norm

$$\|u\|_{H^S(0,T;B)} = \left(\sum_{k=0}^{s} \left\| \frac{\partial^k u}{\partial x^k} \right\|_B^2 \right)^{1/2}$$

we now consider the generalized stability of scheme (5). Suppose that the initial values $u(0), p(0)$,
$u(\tau), p(\tau)$ in (5) have errors $\widetilde{u}_0, \widetilde{p}_0, \widetilde{u}(\tau), \widetilde{p}(\tau)$ and the right hand term in the first and

second equation have errors $\widetilde{f}$ and $\widetilde{g}$ respectively. Then the error $\widetilde{u}(t), \widetilde{p}(t)$ of u(t) and p(t) satisfy

$$\begin{cases} \widetilde{u}_{\hat{t}}(t) + R_\alpha J\left(R_\alpha u(t), \widetilde{u}(t)\right) + R_\alpha J\left(R_\alpha \widetilde{u}(t), u(t) + \widetilde{u}(t)\right) + \nabla \widehat{\widetilde{p}}(t) \\ -\nu \nabla^2 \widehat{\widetilde{u}}(t) = \widetilde{f}(t), \\ p_{\hat{t}}(t) + R_\alpha J\left(R_\alpha p(t), \widetilde{u}(t)\right) + R_\alpha J\left(R_\alpha \widetilde{p}(t), u(t) + \widetilde{u}(t)\right) = \widetilde{g}(t), \\ \widetilde{u}(0) = \widetilde{u}_0, \qquad \widetilde{u}(\tau) = \widetilde{u}_1, \\ \widetilde{p}(0) = \widetilde{p}_0, \qquad \widetilde{p}(\tau) = \widetilde{p}_1, \end{cases} \tag{6}$$

For describing the error, we introduce that $t \in R_\tau$

$$E(t) = \|\widetilde{u}(t)\|^2 + \|\widetilde{p}(t)\|^2 + 2\nu\tau \sum_{t'=\tau}^{t-\tau} \left|\widehat{\widetilde{u}}(t')\right|_1^2,$$

$$\rho(t) = \|\widetilde{u}(0)\|^2 + \|\widetilde{p}(0)\|^2 + \|\widetilde{u}(\tau)\|^2 + \|\widetilde{p}(\tau)\|^2 + \tau \sum_{t'=\tau}^{t-\tau} H(t')$$

$$H(t) = \|\widetilde{f}(t)\|^2 + \|\widetilde{g}(t)\|^2$$

Here after c is positive constant independent of N, τ and any function, which could be different in different cases

Theorem 2.1. *Let τ be suitably small, then there exit positive constants A and A^* depending only on $\nu, |||u|||_\infty$ and $|||p|||_\infty$, such that , if for some $t \in R_\tau$*

$$\rho(t_1)\exp(2At_1) \le \tfrac{A^*}{N},$$

then for all $t \in R_\tau$ and $t \le t_1$, we have

$$E(t) \le \rho(t) e^{2At}$$

Next we consider the convergence. Define

$$U^N = P_N U, \quad P^N = P_N U, \quad \widetilde{U} = U^N - U, \quad \widetilde{P} = P^N - p,$$

we derived from (1) and (5) that

$$\begin{cases} \widetilde{U}_{\hat{t}}(t) + R_\alpha J\left(R_\alpha \widetilde{U}(t), U^N + \widetilde{U}(t)\right) + R_\alpha J\left(R_\alpha U^N, \widetilde{U}\right) - \nu\nabla^2 \widehat{\widetilde{U}}(t) - \nabla \widehat{\widetilde{P}}(t) = \\ \qquad -E_1 - E_2 - E_3 - \nu \triangle E_4 - \nabla E_5, \\ \widetilde{P}_{\hat{t}} + R_\alpha J(R_\alpha \widetilde{P}(t), U^N + \widetilde{U}(t)) + R_\alpha J(R_\alpha P^N, \widetilde{U}) = -E_6 - E_7 - E_8, \\ \widetilde{U}(0) = (P_c - P_N)U_0, \qquad \widetilde{U}(\tau) = (P_c - P_N)(U_0 + \tau\partial_t U(0) - U(\tau)), \\ \widetilde{P}(0) = (P_c - P_N)P_0, \qquad \widetilde{P}(\tau) = (P_c - P_N)(P_0 + \tau\partial_t P(0) - P(\tau)), \end{cases} \tag{7}$$

Where

$$E_1(t) = \frac{\partial U^N}{\partial t} - U^N_{\hat{t}},$$

$$E_2(t) = (R_\alpha P_c - P_N) f$$

$$E_3(t) = R_\alpha J(R_\alpha U^N, U^N) - P_N[(U.\nabla)U],$$

$$E_4(t) = U^N - \widehat{U}^N,$$

$$E_5(t) = \widehat{P}^N - P^N,$$

$$E_6(t) = P^N_{\hat{t}} - \frac{\partial P^N}{\partial t}(t),$$

$$E_7(t) = R_\alpha J(R_\alpha P^N, U^N) - P_N[(U.\nabla)P],$$

We have the following result

Theorem 2.2. *Assume that the exact solution (U,P) of (1) satisfied the following smoothness,* $U \in H^3(0,T;L^2(\Omega)) \bigcap H^2(0,T;H^1(\Omega)) \bigcap C(0,T;H^{\mu+1})$, $P \in H^3(0,T;L^2(\Omega)) \bigcap H^1(0,T;H^1(\Omega)) \bigcap C(0,T;H^{\mu+1})$, $f \in L^2(0,T;H^\mu)$.
Then for all $t \le T$,

$$\|U(t) - u(t)\|^2 \le B^*(\tau^4 + N^{-2s}),$$

Where B^* *is positive constant depending only on* ν *and the norm of U and P in the spaces mentioned above.*

3. The proof of Theorems

We now prove Theorem 2.1. By taking the inner product of the first equation of (5) with $2\widehat{\widetilde{u}}(t)$ and second equation with $2\widehat{\widetilde{P}}(t)$ and combining, we get

$$(\|\widetilde{u}(t)\| + \|\widetilde{p}(t)\|)_{\widehat{t}} + 2\nu|\widehat{\widetilde{u}}(t)|_1^2 + \sum_{j=1}^{5} F_j \le \|\widehat{\widetilde{u}}(t)\|^2 + \|\widehat{\widetilde{p}}(t)\|^2 + H(t) \tag{8}$$

Where

$$F_1 = 2(R_\alpha J(R_\alpha \widetilde{u}(t), \widetilde{u}(t), \widehat{\widetilde{u}}(t))),$$
$$F_2 = 2(R_\alpha J(R_\alpha u(t), \widetilde{u}(t)) + R_\alpha J(R_\alpha \widetilde{u}(t), \widetilde{u}(t), \widehat{\widetilde{u}}(t)),$$
$$F_3 = 2((\nabla \widehat{\widetilde{p}}(t), \widehat{\widetilde{u}}(t)), \widehat{\widetilde{p}}(t)),$$
$$F_4 = 2(R_\alpha J(R_\alpha p(t), \widetilde{u}(t)), \widehat{\widetilde{p}}(t)),$$
$$F_5 = 2(R_\alpha J(R_\alpha \widetilde{p}(t), u(t) + \widetilde{u}(t)), \widehat{\widetilde{p}}(t)),$$
$$H(t) = \|\widetilde{f}(t)\|^2 + \|\widetilde{g}(t)\|^2.$$

Now we are going to estimate $|F_j|$. By using equations (9.1.10), (9.1.15) of [3] and the embedding theorem, we get

$$|F_1| \le \frac{\nu}{2}|\widehat{\widetilde{u}}(t)|_1^2 + \frac{cN}{\nu}\|\widetilde{u}(t)\|^4,$$
$$|F_2| \le \frac{\nu}{2}|\widehat{\widetilde{u}}(t)|_1^2 + \frac{c}{\nu}\||u|\|_\infty^2\|\widetilde{u}(t)\|^2,$$
$$|F_3| \le \frac{\nu}{2}|\widehat{\widetilde{u}}(t)|_1^2 + \frac{2}{\nu}\|\widehat{\widetilde{p}}(t)\|^2,$$
$$|F_4| \le \frac{\nu}{2}|\widehat{\widetilde{u}}(t)|_1^2 + \frac{c}{\nu}\||p|\|_\infty^2\|\widehat{\widetilde{p}}(t)\|^2,$$
$$|F_5| \le \frac{\nu}{2}|\widehat{\widetilde{u}}(t)|_1^2 + \frac{cN}{\nu}\||u|\|_\infty^2(\|\widetilde{p}(t)\|^2 + \|\widehat{\widetilde{p}}(t)\|^2)$$

By substituting the above estimations in (8), we get

$$(\|\widetilde{u}(t)\| + \|\widetilde{p}\|^2)_{\widehat{t}} + \nu|\widehat{\widetilde{u}}|_1^2 \le A(\|\widetilde{u}(t)\|^2 + \|\widetilde{p}(t)\|^2 + \|\widehat{\widetilde{u}}(t)\|^2 + \|\widehat{\widetilde{p}}(t)\|^2) + B\|\widetilde{u}(t)\|^4 + H(t) \tag{9}$$

$$A = 1 + \frac{c}{\nu}(N+1)\||u|\|_\infty^2 + \frac{c}{\nu}\||p|\|_\infty^2 \qquad B = \frac{cN}{\nu},$$

In fact

$$\|\widehat{\widetilde{u}}(t)\|^2 \le \frac{1}{2}(\|\widetilde{u}(t+\tau)\|^2 + \|\widetilde{u}(t-\tau)\|^2),$$

$$\|\widehat{\widetilde{p}}(t)\|^2 \le \frac{1}{2}(\|\widetilde{p}(t+\tau)\|^2 + \|\widetilde{p}(t-\tau)\|^2),$$

Summing up (9) for $t' = \tau, 2\tau, ...t-\tau$, we have

$$E(t) \le \rho(t) + 2\tau \sum_{t'=\tau}^{t-\tau} [AE(t') + BE^2(t')],$$

Where E(t) and $\rho(t)$ are defined in section 2. Finally by applying Lemma 3.2, page 97 of [5], we complete the proof of Theorem 2.1

We now turn to proving Theorem 2.2. we have to estimate the right term in (7)

$$\tau \sum_{t'=\tau}^{t-\tau} \|E_1(t')\|^2 \le c\tau^4 \|U^N\|^2_{H^3(0,T;L^2(\Omega))},$$

$$\tau \sum_{t'=\tau}^{t-\tau} \|E_2(t')\|^2 \le cN^{-2\mu} \|f^N\|^2_{L^2(0,T;H^\mu(\Omega))},$$

$$\tau \sum_{t'=\tau}^{t-\tau} \|E_3(t')\|^2 \le cN^{-2\mu} \||U\||^4_{\mu+1},$$

$$\tau \sum_{t'=\tau}^{t-\tau} |E_4(t')|^2_1 \le c\tau^4 \|U^N\|^2_{H^2(0,T;H^1)},$$

$$\tau \sum_{t'=\tau}^{t-\tau} \|E_5(t')\|^2 \le \tau^4 \|P^N\|^2_{H^2(0,T;L^2)},$$

$$\tau \sum_{t'=\tau}^{t-\tau} \|E_6(t')\|^2 \le c\tau^4 \|P^N\|^2_{H^3(0,T;L^2)},$$

$$\tau \sum_{t'=\tau}^{t-\tau} \|E_7(t')\|^2 \le cN^{-2\mu} \||U\||^4_{\mu+1} + cN^{-2\mu} \||U\||^4_{\mu+1},$$

Finally, we can prove that

$$\|\widetilde{U}(\tau)\|^2 \le c\tau^4 \|U^N\|^2_{H^2(0,T;H^1(\Omega))},$$

$$\|\widetilde{P}(\tau)\|^2 \le c\tau^4 \|P^N\|^2_{H^2(0,T;H^1(\Omega))},$$

$$\|\widetilde{U}(0)\|^2_1 \le cN^{-2\mu} \|U_0\|^2_{\mu+1},$$

$$\|\widetilde{P}(0)\|^2_1 \le cN^{-2\mu} \|P_0\|^2_{\mu+1},$$

Thus by the argument similar to that in the proof of Theorem 2.1, we get

$$(\|\widetilde{U}(t)\|^2 + \|\widetilde{P}(t)\|^2)_{\hat{t}} + \nu |\widehat{\widetilde{U}}(t)|^2_1 \le A_1(\|\widetilde{U}(t)\|^2 + \|\widetilde{P}(t)\|^2 + \|\widehat{\widetilde{U}}(t)\|^2 + \|\widehat{\widetilde{P}}(t)\|^2) + B_1 \|\widehat{\widetilde{U}}(t)\|^4 + H_1(t) \tag{10}$$

$$A_1 = 1 + \frac{c}{\nu}(N+1)\||U\||^2_{1,\infty} + \frac{c}{\nu}\||P\||^2_{1,\infty} \qquad B_1 = B,$$

$$H_1(t) = A_2(\tau^4 + N^{-2s})$$

$$E(t) = \|\widetilde{U}(t)\|^2 + \|\widetilde{P}(t)\|^2 + 2\nu\tau \sum_{t_\prime=\tau}^{t-\tau} |\widetilde{U}(t')|^2_1,$$

$$\rho(t) = \|\widetilde{U}(0)\|^2 + \|\widetilde{P}(0)\|^2 + \|\widetilde{U}(\tau)\|^2 + \|\widetilde{P}(\tau)\|^2 + \tau \sum_{t'=\tau}^{t-\tau} H_1(t') \le B_2(\tau^4 + N^{-2s}),$$

Where A_1 and B_2 are positive constants depending on ν and the norm of U and P in the spaces mentioned above. Finally by summing up (10) for $t \in R_\tau$, and applying Lemma 3.2 page 97 of [5], we complete the proof of Theorem 2.2.

4. Numerical Results

The test functions are

$$\begin{cases} U_1(x_1, x_2, t) = -\cos(x_1)\sin(x_2)\exp(wt) \\ U_2(x_1, x_2, t) = \sin(x_1)\cos(x_2)\exp(wt) \\ P(x_1, x_2) = \cos(x_1)\sin(x_2) \end{cases} \tag{11}$$

and

$$\begin{cases} U_1(x_1, x_2, t) = -A\cos(x_2)\exp(\sin(x_1) + \sin(x_2) + wt) \\ U_2(x_1, x_2, t) = A\cos(x_1)\exp(\sin(x_1) + \sin(x_2) + wt) \\ P(x_1, x_2) = A\exp(\sin(x_1) + \sin(x_2)) \end{cases} \tag{12}$$

where A and w are parameters. For describing the errors, we define

$$E(U_i(t)) = \left[\frac{\sum_{x\in\Omega_N} |U_i(t) - u_i(t)|^2}{\sum_{x\in\Omega_N |U_i(t)|^2}}\right]^{1/2}, \qquad i = 1, 2 \tag{13}$$

$$E(P(t)) = \left[\frac{\sum_{x\in\Omega_N} |P(t) - P(t)|^2}{\sum_{x\in\Omega_N |P(t)|^2}}\right]^{1/2}, \tag{14}$$

where u_i and p are the solution of scheme (15). For comparison, we consider also two level pseudospectral scheme of (1) (see [9]). The calculation is carried out for $\tau = 0.01$.
We first consider test function (11), the results shows that the computation is quite accurate, even with small N (N=4). Besides the three-level scheme gives better results than the two-level one and the effect of the restraint operator is not very clear (see Table 1 and Table 2).
Secondly, we examine the effect of R_α on the computation by test function (12), if the vibration of the genuine solution of (1) is small (e.g A=0.1), the effect of R_α is not clear (see Table 3 and 4).

If the vibration of genuine solution is big with small viscosity, (e.g., A=1.0, v=.001), then the effect of R_α is very clear (see Table 5).
The value of α in the restraint operator must be suitably chosen. if α is too small, the approximation accuracy is lowered. The best vale of α is different in different cases. In this computation suitable choice of α is about 5 as in [8].

Table 1. $N = 4, \quad \nu = 0.001, \quad w = -0.02, \quad t = 1.0$

	Three-Level Scheme(5)			Two-Level Scheme		
	$\alpha = 5$	$\alpha = 10$	$\alpha = \infty$	$\alpha = 5$	$\alpha = 10$	$\alpha = \infty$
$E(u_1, t)$	.11277E-5	.16131E-5	.17347E-5	.62866E-4	.61588E-4	.61590E-4
$E(u_2, t)$	.11265E-5	.15695E-5	.16654E-5	.60811E-4	.63040E-4	.63014E-4
$E(p, t)$	.97947E-6	.92526E-6	.16099E-5	.10042E-3	.11388E-4	.65148E-4

TABLE 2. $N = 4$, $\nu = 0.00001$, $w = 1.00$, $t = 1.0$

	Three-Level Scheme(5)			Two-Level Scheme		
	$\alpha = 5$	$\alpha = 10$	$\alpha = \infty$	$\alpha = 5$	$\alpha = 10$	$\alpha = \infty$
$E(u_1, t)$	.40200E-3	.39606E-3	.39669E-3	.40614E-2	.39749E-2	.39815E-2
$E(u_2, t)$	.40200E-3	.39605E-3	.39668E-3	.40608E-2	.39743E-2	.39805E-2
$E(p, t)$	.17225E-3	.17687E-3	.17929E-3	.17459E-2	.17777E-2	.18108E-2

TABLE 3. $N = 8$, $\nu = 0.01$, $A = 0.1$ $w = 0.1$, $t = 1.0$

	Three-Level Scheme(5)			Two-Level Scheme		
	$\alpha = 5$	$\alpha = 10$	$\alpha = \infty$	$\alpha = 5$	$\alpha = 10$	$\alpha = \infty$
$E(u_1, t)$	.38458E-4	.36323E-4	.36323E-4	.58680E-4	.55130E-4	.55094E-4
$E(u_2, t)$	.38410E-4	.36323E-4	.36331E-4	.57759E-4	.54549E-4	.54512E-4
$E(p, t)$	.20745E-4	.15669E-4	.15626E-5	.86585E-5	.79643E-5	.79752E-5

TABLE 4. $N = 8$, $\nu = 0.1$, $A = 0.1$ $w = 0.1$, $t = 1.0$

	Three-Level Scheme(5)			Two-Level Scheme		
	$\alpha = 5$	$\alpha = 10$	$\alpha = \infty$	$\alpha = 5$	$\alpha = 10$	$\alpha = \infty$
$E(u_1, t)$	.26178E-3	.27205E-3	.28229E-3	.30902E-3	.31767E-3	.32872E-3
$E(u_2, t)$	.26186E-3	.27047E-3	.28199E-3	.30910E-3	.31717E-3	.32925E-3
$E(p, t)$	.19092E-3	.20182E-3	.21099E-3	.21319E-3	.22309E-3	.23246E-3

TABLE 5. $N = 8$, $\nu = 0.001$, $A = 1.0$ $w = 0.1$, $t = 1.0$

	Three-Level Scheme(5)			Two-Level Scheme		
	$\alpha = 5$	$\alpha = 10$	$\alpha = \infty$	$\alpha = 5$	$\alpha = 10$	$\alpha = \infty$
$E(u_1, t)$	.49588E-3	.36755E-2	.26314E-1	.56969E-3	.35156E-2	.12100E+0
$E(u_2, t)$	.45328E-3	.27603E-2	.26202E-1	.52023E-3	.28866E-2	.13685E+0
$E(p, t)$	.32006E-3	.22852E-2	.19102E-1	.37414E-3	.22331E-2	.72320E-1

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