MATH 231. EXAM 3

Name:____

Instructions: This is a closed-book exam, and calculators can only be used to do basic arithmetic operations (not allowed for differentiations, integrations, Laplace transforms, or inverse Laplace transforms). The last page contains a table for integrals and some results from textbook, which might be helpful. Read each problem carefully. You must show your work to receive credit. Partial credit will be given for any work relevant to the problem.

Problem	**Grade**
1	
2	
3	
4	
5	
Total	

There is a total of 100 points.

Problem 1 (15 points): (a) (5 pts) Compute the Laplace transform $\mathcal{L}\left[e^{-3t}t^2 - 4\right]$.

(b) (10 pts) Compute $\mathcal{L}[f(t)]$ where

$$f(t) = \begin{cases} 0, & 0 < t < 1, \\ t+1, & 1 < t < 2, \\ 0, & t > 2. \end{cases}$$

Problem 2 (15 points): (a) (7 pts) Compute the inverse Laplace transform $\mathcal{L}^{-1}\left[\dfrac{s}{s^2-4s+5}\right]$.

(b) (8 points) Compute $\mathcal{L}^{-1}[F(s)]$ where $F(s) = \dfrac{4s-2}{s^2-1}$.

Problem 3 (20 points): Use the **Laplace transform** to solve the IVP:

$$y'' + 4y = 4 \quad \text{with} \quad y(0) = 0, y'(0) = 1.$$

Problem 4 (25 points): Solve the IVP:

$$y' + 2y = f(t) \quad \text{with} \quad y(0) = 1,$$

where

$$f(t) = \begin{cases} 0, & 0 < t < 1, \\ 2t, & t > 1. \end{cases}$$

Problem 5 (25 points): Find the recurrence relation and the first four nonzero terms in a power series expansion about $x = 0$ of the solution to the following IVP:

$$y'' + xy' - (x-1)y = 0 \quad \text{with} \quad y(0) = 6,\ y'(0) = -3.$$

Table of Laplace Transforms

Function	Example
$\mathcal{L}[0]=0$	N/A
$\mathcal{L}[c]=\frac{c}{s}$	$\mathcal{L}[231]=\frac{231}{s}$
$\mathcal{L}[t^n]=\frac{n!}{s^{n+1}}$	$\mathcal{L}[t^3]=\frac{3!}{s^4}$
$\mathcal{L}[e^{at}]=\frac{1}{s-a}$	$\mathcal{L}[e^{5t}]=\frac{1}{s-5}$
$\mathcal{L}[e^{at}t^n]=\frac{n!}{(s-a)^{n+1}}$	$\mathcal{L}[e^{2t}t^4]=\frac{4!}{(s-2)^5}$
$\mathcal{L}[\cos(bt)]=\frac{s}{s^2+b^2}$	$\mathcal{L}[\cos(7t)]=\frac{s}{s^2+49}$
$\mathcal{L}[\sin(bt)]=\frac{b}{s^2+b^2}$	$\mathcal{L}[\sin(7t)]=\frac{7}{s^2+49}$
$\mathcal{L}[e^{at}\cos(bt)]=\frac{s-a}{(s-a)^2+b^2}$	$\mathcal{L}[e^{5t}\cos(3t)]=\frac{s-5}{(s-5)^2+9}$
$\mathcal{L}[e^{at}\sin(bt)]=\frac{b}{(s-a)^2+b^2}$	$\mathcal{L}[e^{5t}\sin(3t)]=\frac{3}{(s-5)^2+9}$
$\mathcal{L}[af(t)+bg(t)]=a\mathcal{L}[f(t)]+b\mathcal{L}[g(t)]$	$\mathcal{L}[2+5t]=\mathcal{L}[2]+5\mathcal{L}[t]=\frac{2}{s}+5\left(\frac{1}{s^2}\right)$

Properties of Laplace Transforms

$\mathcal{L}[e^{at}f(t)](s)=\mathcal{L}[f](s-a).$

$\mathcal{L}[t^n f(t)](s)=(-1)^n\dfrac{d^n}{ds^n}\left(\mathcal{L}[f](s)\right).$

$\mathcal{L}[y'](s)=s\mathcal{L}[y](s)-f(0).$

$\mathcal{L}[y''](s)=s^2\mathcal{L}[y](s)-sy(0)-y'(0).$

$\mathcal{L}[f(t-a)u(t-a)](s)=e^{-as}F(s),\ \text{where } F(s)=\mathcal{L}[f(t)],$

$\mathcal{L}[g(t)u(t-a)](s)=e^{-as}\mathcal{L}[g(t+a)](s)$

$\mathcal{L}^{-1}[e^{-as}F(s)]=f(t-a)u(t-a),\ \text{where } f(t)=\mathcal{L}^{-1}[F(s)]$

Table for Integrals:

$\int\frac{dx}{\sqrt{x^2+a^2}}=\ln|x+\sqrt{x^2+a^2}|.$

$\int\frac{dx}{\sqrt{x^2-a^2}}=\ln|x+\sqrt{x^2-a^2}|,\quad x^2\geq a^2.$

$\int\frac{dx}{\sqrt{a^2-x^2}}=\arcsin(\frac{x}{a}),\quad a^2\geq x^2.$

$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\arctan(\frac{x}{a}).$

$\int\tan(x)dx=-\ln|\cos x|.$

$\int\cot(x)dx=\ln|\sin x|.$

$\int\sec(x)dx=\ln|\sec x+\tan x|.$

$\int\csc(x)dx=-\ln|\csc x+\cot x|.$

Common trigonometric substitutions:

(1) For integrand involving $\sqrt{a^2-x^2}$, set $x=a\sin(\theta)$,
(2) For integrand involving $\sqrt{a^2+x^2}$, set $x=a\tan(\theta)$,
(3) For integrand involving $\sqrt{x^2-a^2}$, set $x=a\sec(\theta)$,
(4) For $\int\tan^n(x)\sec^{2m}(x)dx$, set $u=\tan(x)$,
(5) For $\int\cot^n(x)\csc^{2m}(x)dx$, set $u=\cot(x)$.