MATH 231. EXAM 1

Name:_____

Instructions: This is a closed-book exam, and calculators can only be used to do basic arithmetic operations (not allowed for differentiation and integration). The last page contains a table for integrals and some results from textbook, which might be helpful. Read each problem carefully. You must show your work to receive credit. Partial credit will be given for any work relevant to the problem.

Problem	Grade
1	
2	
3	
4	
5	
Total	

There is a total of 100 points.

Problem 1 (10 points): (a) (5 points) Check whether $y = e^{2t} + 2t$ is a solution to the differential equation

$$y'' - 2y' = 0$$

or not. Show your reasoning.

(b) (5 points) Check whether the following equation

$$(e^x y^2 + 3x^2 y)dx - (2e^{-x} y - x^3)dy = 0$$

is exact or not. Show your reasoning.

Problem 2 (20 points): Solve the following ODE:

$$y(y+1)\frac{dy}{dx} = y(y-2)^2 \sin(x).$$

Problem 3 (20 points): Solve the following initial value problem:

$$x\frac{dy}{dx} = y + \frac{x^2y^2}{2y^3 + x^3}, \qquad y(1) = 2.$$

Problem 4 (25 points): Consider the following equation:

$$(-1 - y)\, dx + (2x^2 y + x)\, dy = 0.$$

Use the **Theorem for Special Integrating Factors** on the last page to find an integrating factor and then solve the equation.

Problem 5 (25 points): A 10L tank initially contains 4L salt water at a concentration of 2.5 g/L. Salt water at a concentration of 0.5 g/L is being pumped in at 4 L/min and the mixture is leaving the tank at 2 L/min. Compute the amount of salt in the tank when the tank is full.

Brief Table for Integrals:

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x+\sqrt{x^2+a^2}|. \qquad \int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}|, \quad x^2 \geq a^2.$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin(\tfrac{x}{a}), \quad a^2 \geq x^2. \qquad \int \frac{dx}{a^2+x^2} = \tfrac{1}{a}\arctan(\tfrac{x}{a}).$$

Common trigonometric substitutions:

(1) For integrand involving $\sqrt{a^2-x^2}$, set $x = a\sin(\theta)$,
(2) For integrand involving $\sqrt{a^2+x^2}$, set $x = a\tan(\theta)$,
(3) For integrand involving $\sqrt{x^2-a^2}$, set $x = a\sec(\theta)$,
(4) For $\int \tan^n(x)\sec^{2m}(x)dx$, set $u = \tan(x)$,
(5) For $\int \cot^n(x)\csc^{2m}(x)dx$, set $u = \cot(x)$.

Theorem for Special Integrating Factors. Consider the equation

$$M(x,y)dx + N(x,y)dy = 0, \tag{1}$$

if $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu(x) = \exp\left[\int \left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right) dx\right]$$

is an integrating factor for equation (1);
if $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on y, then

$$\mu(y) = \exp\left[\int \left(\frac{\partial N/\partial x - \partial M/\partial y}{M}\right) dy\right]$$

is an integrating factor for equation (1).