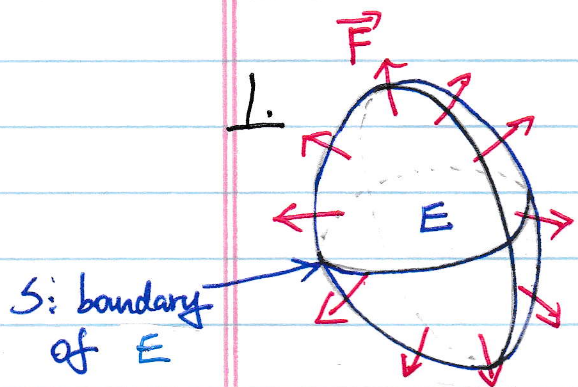
16.9 The Divergence Theorem (Gauss' Theorem)

1.



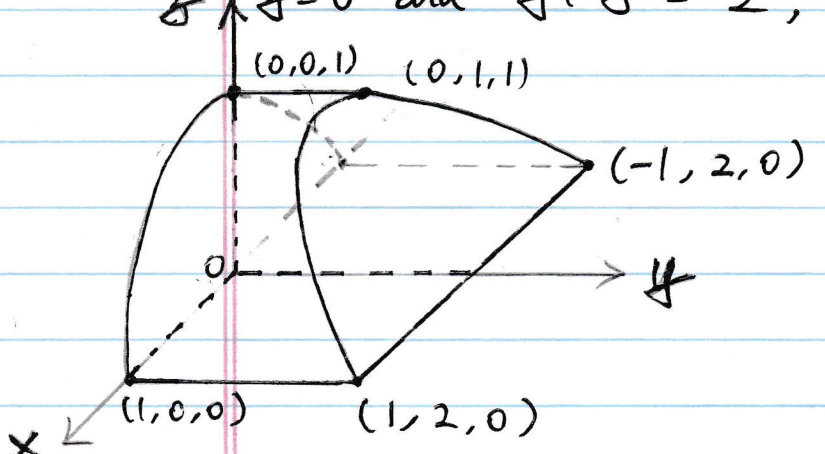
Suppose that S is the surface completely surrounds a solid region E and S has outwards orientation. Then:

$$\underbrace{\iint_S \vec{F}\cdot\vec{n}\, dS}_{\text{Flux over (across) the surface } S} = \iiint_E \nabla\cdot\vec{F}\, dV$$

Comments: (1) S is the surface, E is the solid inside S.
(2) How we do $\iiint_E \nabla\cdot\vec{F}\, dV$ depends on E. It could be rectangular, cylindrical or spherical.

Text-Example 2 Evaluate $\iint_S \vec{F}\cdot\vec{n}\, dS$ where

$\vec{F}(x,y,z) = \langle xy,\ y^2 + e^{xz^2},\ \sin(xy)\rangle$ and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$ and $y + z = 2$, and S has outwards orientation.



Solution: Solid region E is described as

$$-1 \le x \le 1,\quad 0 \le z \le 1 - x^2,$$
$$0 \le y \le 2 - z$$

Also, $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 + e^{xz^2}) + \frac{\partial}{\partial z}(\sin(xy))$

$$= y + 2y + 0 = 3y.$$

So by the divergence theorem,

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_E \nabla \cdot \vec{F}\, dV$$

$$= \iiint_E 3y\, dV$$

$$= \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y\, dy\, dz\, dx$$

$$= \int_{-1}^{1} \int_0^{1-x^2} \left[\frac{3}{2} y^2\right]_{y=0}^{y=2-z} dz\, dx$$

$$= \int_{-1}^{1} \int_0^{1-x^2} \frac{3}{2}(2-z)^2\, dz\, dx$$

$$= \int_{-1}^{1} \left[-\frac{3}{2}\frac{1}{3}(2-z)^3\right]_{z=0}^{z=1-x^2} dx$$

$$= \int_{-1}^{1} 4 - \frac{1}{2}(x^2+1)^3\, dx$$

$$= \int_{-1}^{1} -\frac{1}{2}x^6 - \frac{3}{2}x^4 - \frac{3}{2}x^2 + \frac{7}{2}\, dx$$

$$= \frac{184}{35}.$$

Example: Suppose S is the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 5$ with the caps at the ends, oriented outwards. Evaluate $\iint_S (x^2 \vec{i} + xz\, \vec{j} + z\, \vec{k}) \cdot \vec{n}\, dS$

Solution: Let E be the solid region bounded by S. Then by the divergence theorem

$$\iint_S \langle x^2, xz, z\rangle \cdot \vec{n}\, dS$$

$$= \iiint_E \left(\nabla \cdot \langle x^2, xz, z\rangle\right) dV$$

$$= \iiint_E 2x + 0 + 1 \; dV = \iiint_E (2x+1)\, dV$$

The solid region E is described in cylindrical coordinates as:

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 2, \quad 0 \le z \le 5.$$

(Note that $x^2+y^2=4$ becomes $r=2$ in cylindrical coordinates)

Hence $\iiint_E (2x+1)\, dV$

$$= \int_0^{2\pi}\int_0^2\int_0^5 (2r\cos(\theta)+1)\, r \; dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 10r^2\cos(\theta) + 5r \; dr\, d\theta$$

$$= \int_0^{2\pi} \frac{80}{3}\cos(\theta) + 10 \; d\theta$$

$$= 20\pi.$$

2. Notes

(1) If S is oriented inwards, then we need to negate.

(2) If $\nabla \cdot \vec{F}$ is a constant, then $\iiint_E \nabla \cdot \vec{F}\, dV$

is the constant times the volume of E. This is only useful if the volume can be conveniently calculated.

(3) S must completely surround E in order to use the divergence theorem.

"Text-Ex 1" Find the flux of the vector field $\vec{F}(x,y,z) = z\vec{i} + y\vec{j} + x\vec{k}$ over the surface S: $x^2+y^2+z^2=1$, oriented inwards.

Solution: Since the orientation is inward, we have

$$\iint_S \vec{F}\cdot\vec{n}\,dS = -\iiint_E \nabla\cdot\vec{F}\,dV$$

$$= -\iiint_E \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x)\,dV$$

$$= -\iiint_E 1\,dV$$

$$= \underbrace{-1\cdot \text{Volume}(E)}$$

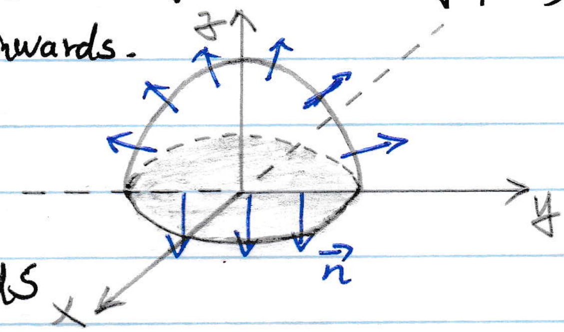
$$= -1\cdot\frac{4\pi}{3} = -\frac{4\pi}{3}.$$

Negative sign. See note (1).

Here E is the unit sphere bounded by S.

We are using note (2) because the $\nabla\cdot\vec{F}$ is equal to constant 1.

Example: Evaluate $\iint_S \left(2x\,\vec{i} + 5y\,\vec{j} + 7z\,\vec{k}\right)\cdot\vec{n}\,dS$ where S is the top hemisphere (the half above the xy-plane) of $x^2+y^2+z^2=9$, along with the base, oriented inwards.



Solution: By the divergence theorem.

$$\iint_S \left(2x\,\vec{i} + 5y\,\vec{j} + 7z\,\vec{k}\right)\cdot\vec{n}\,dS$$

$$= -\iiint_E \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(5y) + \frac{\partial}{\partial z}(7z)\,dV$$

$$= -\iiint_E 2+5+7\,dV$$

$$= -\iiint_E 14\,dV$$

$$= -14\ \text{Volume}(E)$$

b/c E is the hemisphere

$$= -14\cdot\frac{1}{2}\,\underbrace{\left(\frac{4\pi}{3}\right)(3)^3}$$

Volume of sphere $= \frac{4\pi}{3}R^3$

$$= -252\pi$$

Remark: If S does not contain the base, then we cannot use the divergence theorem. We have to use the parametrization of S. Using spherical coordinates, we see $x^2+y^2+z^2=9$ becomes $\rho=3$.

Parametrization of S: $\vec{r}(\theta,\varphi) = \rho\sin(\phi)\cos(\theta)\,\vec{i} + \rho\sin(\phi)\sin(\theta)\,\vec{j} + \rho\cos(\phi)\,\vec{k}$

$$= 3\sin(\phi)\cos(\theta)\,\vec{i} + 3\sin(\phi)\sin(\theta)\,\vec{j} + 3\cos(\phi)\,\vec{k}$$

$$0 \le \phi \le \underset{\uparrow}{\frac{\pi}{2}}, \quad 0 \le \theta \le 2\pi.$$

b/c the hemisphere is above the xy-plane.

$$\vec{r}_\theta = \langle -3\sin(\phi)\sin(\theta),\ 3\sin(\phi)\cos(\theta),\ 0 \rangle$$

$$\vec{r}_\phi = \langle 3\cos(\phi)\cos(\theta),\ 3\cos(\phi)\sin(\theta),\ -3\sin(\phi) \rangle$$

$$\vec{r}_\theta \times \vec{r}_\phi = \langle -9\sin^2(\phi)\cos(\theta),\ -9\sin^2(\phi)\sin(\theta),\ -9\sin(\phi)\cos(\phi) \rangle$$

$$= \underbrace{-9\sin(\phi)}_{<0} \underbrace{\langle \sin(\phi)\cos(\theta),\ \sin(\phi)\sin(\theta),\ \cos(\phi) \rangle}_{\text{Same direction as } \overrightarrow{OP} \text{ where } P \text{ is a point on the sphere.}}$$

So $\vec{r}_\theta \times \vec{r}_\phi$ does not match the orientation of S.

$$\iint_S \left(2x\,\vec{i} + 5y\,\vec{j} + 7z\,\vec{k}\right) \cdot \vec{n}\, dS$$

$$= -\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \langle 2(3\sin(\phi)\cos(\theta)),\ 5(3\sin(\phi)\sin(\theta)),\ 7(3\cos(\phi)) \rangle$$

b/c the orientation does not match

$$\cdot \left(-9\sin(\phi)\right) \langle \sin(\phi)\cos(\theta),\ \sin(\phi)\sin(\theta),\ \cos(\phi) \rangle \, d\theta\, d\phi$$

$= \cdots$

3. Source and sink

If $\operatorname{div} \vec{F}(P) > 0$, then P is called a source;

if $\operatorname{div} \vec{F}(P) < 0$, then P is called a sink.