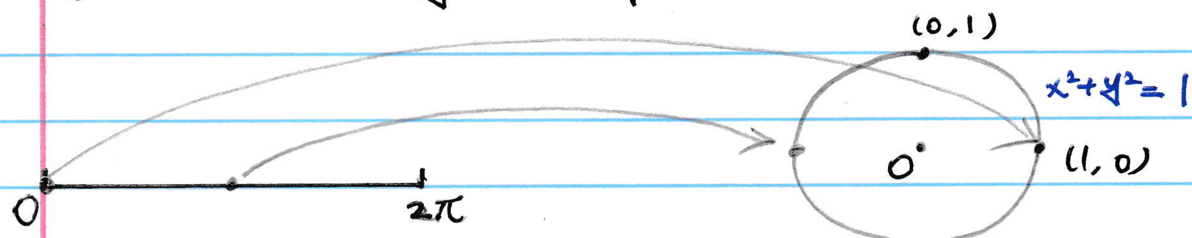
16.6 Parametric Surfaces and Their Areas

1. Introduction of parametrization of surfaces

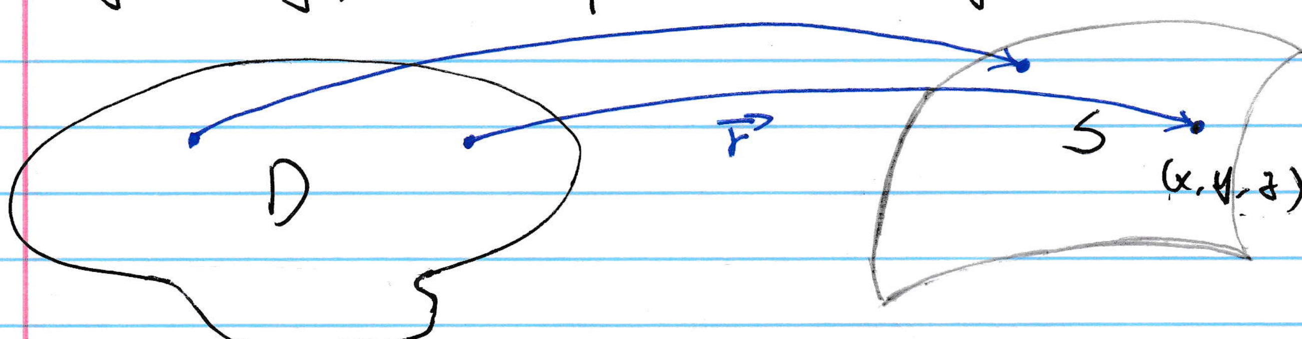
Recall how we parametrized a curve $\vec{r}(t)$: for various t we think of $\vec{r}(t)$ as a point on the curve and for all t we get all points on the curve.



$$\vec{r}(t) = \cos(t)\,\vec{i} + \sin(t)\,\vec{j} \qquad 0 \le t \le 2\pi$$

Range of t is an interval, range of $\vec{r}(t)$ is the curve.

- Idea of parametrizations of a given surface S.
  We want a parametrization $\vec{r}(u, v)$ for a range of u and v so that as those variables run over their ranges we get all the points on the surface.



$D \subseteq \mathbb{R}^2$ range of (u, v)

range of $\vec{r}(u, v)$ is the surface $S \subseteq \mathbb{R}^3$.

$$\vec{r}(t,s) = \langle 3\cos(t) + \cos(t)\cos(s),\ 3\sin(t) + \sin(t)\cos(s),\ \sin(s) \rangle$$

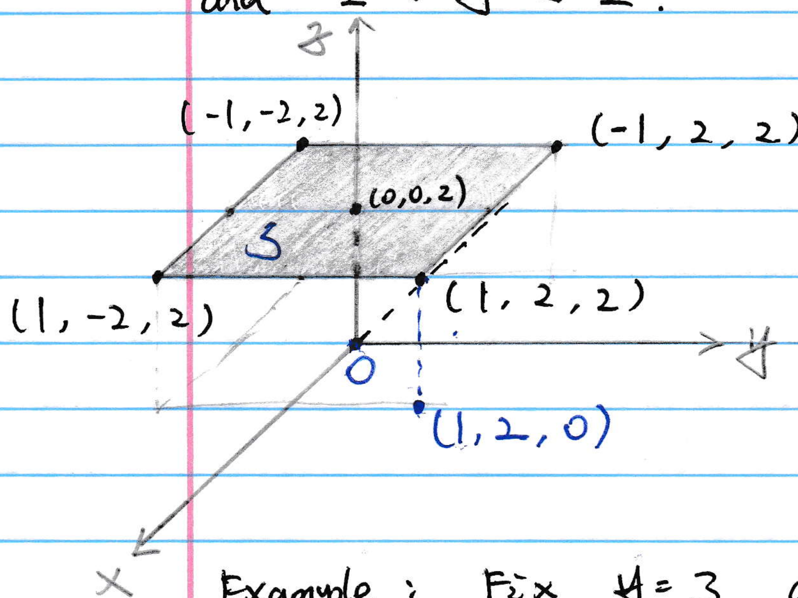
- Watch online video.
- Example using a sheet of paper.

Sometimes we don't use u and v, other common variables are $x, y, z, r, \theta, \phi, \rho$.

2. Examples:

Notice that the choice of the two variables (u, v) is tricky and confusing at first. The choice is based on the restriction or the surface.

Example: A small rectangle at $z = 2$ with $-1 \le x \le 1$ and $-2 \le y \le 2$.



Parametrization

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + 2\vec{k}$$

$$\underbrace{-1 \le x \le 1,\quad -2 \le y \le 2}$$

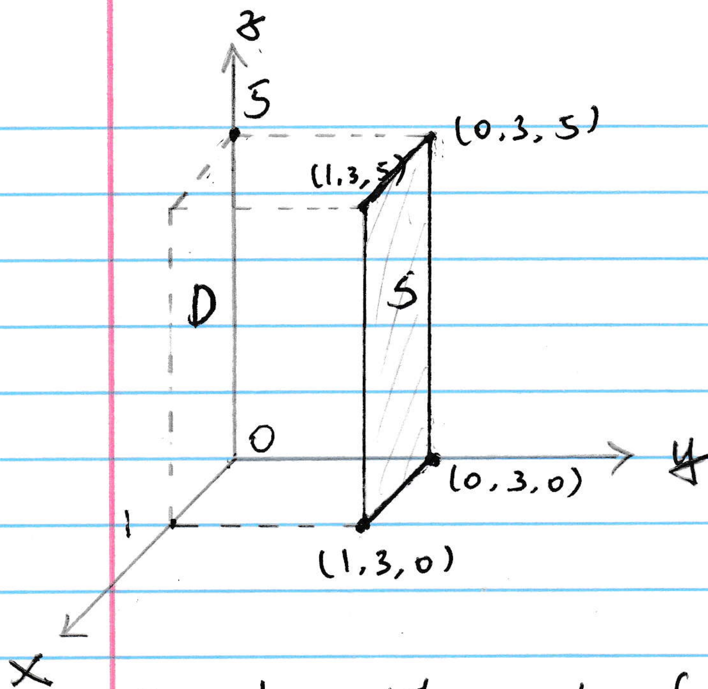
This defines a rectangular region D.

Example: Fix $y = 3$ and consider a parametrization

$$\vec{r}(x,z) = x\vec{i} + 3\vec{j} + z\vec{k}$$

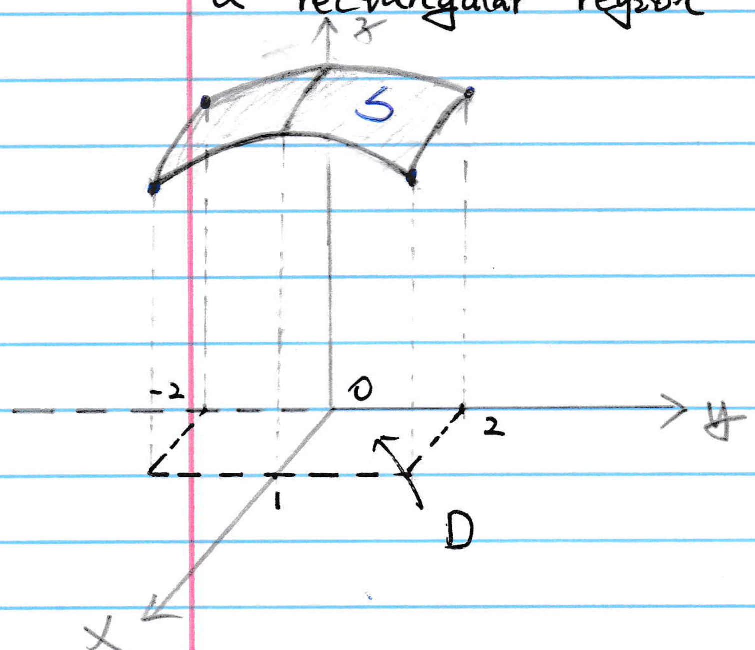
$$= \langle x, 3, z \rangle$$

$0 \le x \le 1$ and $0 \le z \le 5$.



S is the rectangular surface in $\mathbb{R}^3$ with corners $(1,3,0)$, $(0,3,0)$, $(0,3,5)$ and $(1,3,5)$.

<u>Example</u>: The part of $z = \underbrace{9 - x^2 - y^2}_{f(x,y)}$ above a rectangular region $D = \{(x,y) : 0 \le x \le 1, -2 \le y \le 2\}$.



Parametrization

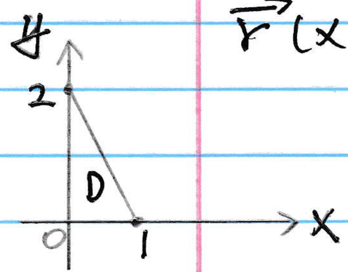
$$\vec{r}(x,y) = x\,\vec{i} + y\,\vec{j} + (9-x^2-y^2)\,\vec{k}$$

$$0 \le x \le 1, \quad -2 \le y \le 2.$$

① In fact, if S is a part of the graph of a function $z = f(x,y)$ defined on some x, y which are themselves nicely parametrized by rectangular coordinates, then we can use
$$\vec{r}(x,y) = x\,\vec{i} + y\,\vec{j} + f(x,y)\,\vec{k}$$
with D the region of allowable x and y.

Example: The part of $z = 9 - x^2 - y^2$ above a triangular region D in the xy-plane with corners $(0,0)$, $(1,0)$, $(0,2)$.

Parametrization:



$$\vec{r}(x, y) = x\,\vec{i} + y\,\vec{j} + (9 - x^2 - y^2)\,\vec{k}$$

$$0 \le x \le 1, \quad 0 \le y \le 2 - 2x.$$

② If S is part of $z = f(x, y)$ and the region of x, y can be parameterized nicely by polar coordinates. Then we use r, θ as parameters

$$\vec{r}(r, \theta) = \underbrace{r\cos(\theta)}_{= x}\,\vec{i} + \underbrace{r\sin(\theta)}_{= y}\,\vec{j} + f(r\cos(\theta), r\sin(\theta))\,\vec{k}$$

and some inequalities for r, θ

Example The disk of radius 2 at $z = 3$ centered on the z-axis.

Parametrization: We know $z = 3$, $x^2 + y^2 \le 2^2$.

Use polar coordinates r, θ for $x^2 + y^2 \le 4$:

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 2.$$

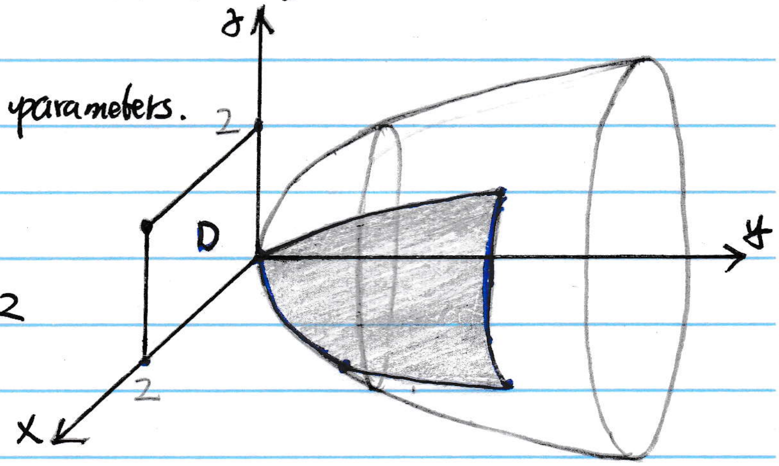
Hence $$\vec{r}(r, \theta) = r\cos(\theta)\,\vec{i} + r\sin(\theta)\,\vec{j} + 3\,\vec{k}$$

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 2.$$

③ Cases ① and ② for S as part of $x = f(y, z)$ or $y = f(x, z)$.

Example : S is part of the paraboloid $y = x^2 + z^2$ to the right of the square in the xz-plane with corners $(x, z) = (0,0), (2,0), (0,2), (2,2)$.

Parametrization : Use x, z as parameters.



$$\vec{r}(x, z) = x\vec{i} + (x^2 + z^2)\vec{j} + z\vec{k}$$

$$0 \le x \le 2, \quad 0 \le z \le 2$$

④ Use cylindrical coordinates or spherical coordinates. Notice that you need to choose which two variables to be used as parameters.

Example : Cylinder $x^2 + y^2 = 9$ between $z = 0$ and $z = 5$.

Parametrization : In cylindrical coordinates $x^2 + y^2 = 9$ is:

$r = 3$. We also have $0 \le z \le 5$ from the problem. Since r is fixed, we use θ, z as our parameters.

$$\vec{r}(\theta, z) = 3\cos(\theta)\,\vec{i} + 3\sin(\theta)\,\vec{j} + z\vec{k}$$

$$0 \le \theta \le 2\pi, \quad 0 \le z \le 5.$$

Text-Ex 4 Find the parametrization of $x^2+y^2+z^2=a^2$ for given $a>0$.

Solution: Use polar coordinates. $x^2+y^2+z^2=a^2$ becomes

$$\rho = a.$$

No other restrictions. So for θ, φ:

$$0 \le \theta \le 2\pi, \quad 0 \le \varphi \le \pi.$$

We have the parametrization:

$$\vec{r}(\theta,\varphi) = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

$$= \rho\sin(\varphi)\cos(\theta)\,\vec{i} + \rho\sin(\varphi)\sin(\theta)\,\vec{j} + \rho\cos(\varphi)\,\vec{k}$$

$$= a\sin(\varphi)\cos(\theta)\,\vec{i} + a\sin(\varphi)\sin(\theta)\,\vec{j} + a\cos(\varphi)\,\vec{k}.$$

Hence $\vec{r}(\theta,\varphi) = a\sin(\varphi)\cos(\theta)\,\vec{i} + a\sin(\varphi)\sin(\theta)\,\vec{j} + a\cos(\varphi)\,\vec{k}$

$0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$.

Another example for ②: S is part of the cone $z = 2+\sqrt{x^2+y^2}$ inside the cylinder $x^2+y^2=4$.

Solution: $x^2+y^2 \le 4$. In polar this is

$0 \le r \le 2$, $0 \le \theta \le 2\pi$.

Use r, θ as parameters, rewrite

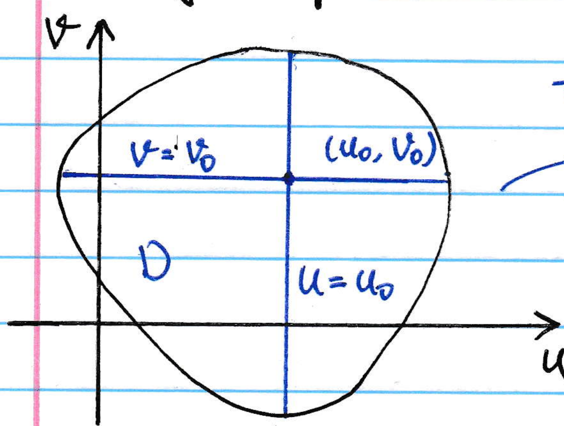
$$z = 2+\sqrt{x^2+y^2} = 2+r.$$

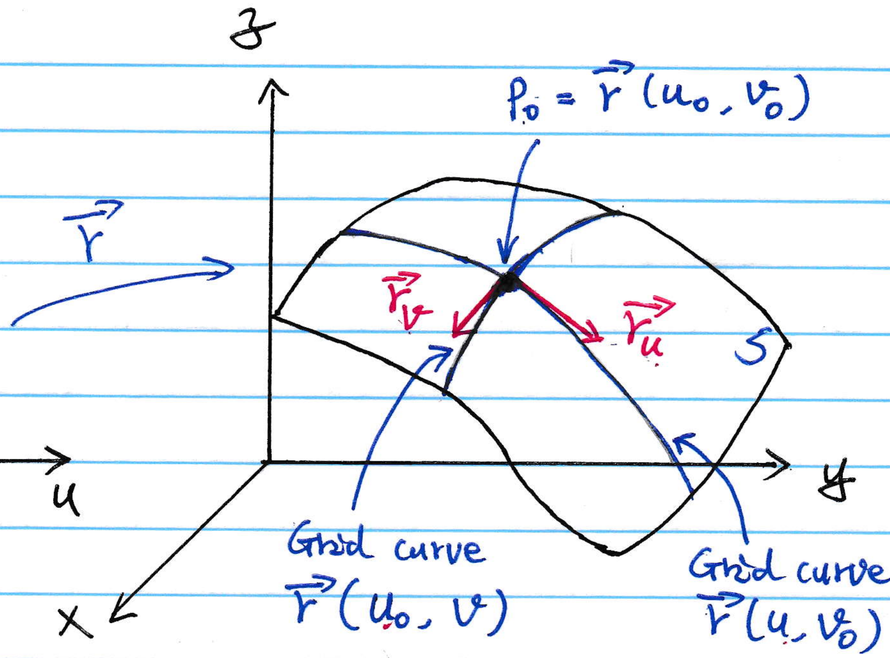
So the parametrization is

$$\vec{r}(r,\theta) = r\cos(\theta)\,\vec{i} + r\sin(\theta)\,\vec{j} + (2+r)\,\vec{k},$$

$0 \le r \le 2$, $0 \le \theta \le 2\pi$

3. Tangent planes





Suppose $r(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$, then

$$\vec{r}_u(u, v) = \frac{\partial x}{\partial u}(u, v)\,\vec{i} + \frac{\partial y}{\partial u}(u, v)\,\vec{j} + \frac{\partial z}{\partial u}(u, v)\,\vec{k}$$

$$\vec{r}_v(u, v) = \frac{\partial x}{\partial v}(u, v)\,\vec{i} + \frac{\partial y}{\partial v}(u, v)\,\vec{j} + \frac{\partial z}{\partial v}(u, v)\,\vec{k}$$

From the picture, $\vec{r}_u(u_0, v_0)$, $\vec{r}_v(u_0, v_0)$ are two vectors on the tangent plane at $P_0 = \vec{r}(u_0, v_0)$. So the normal vector of the tangent plane at P_0 is

$$\vec{n} = \vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0)$$

Equation of the tangent plane is then given by

$$\langle x, y, z\rangle \cdot \vec{n} = \vec{r}(u_0, v_0) \cdot \vec{n}$$

or $$\Big(\langle x, y, z\rangle - \vec{r}(u_0, v_0)\Big) \cdot \vec{n} = 0$$

Text - Ex 9 Find the tangent plane to the surface with parametrization

$$\vec{r}(u, v) = u^2 \vec{i} + v^2 \vec{j} + (u + 2v) \vec{k}$$

at the point $\vec{r}(1, 1) = (1, 1, 3)$.

Solution: $\vec{r}_u(u, v) = \left\langle \frac{\partial}{\partial u}(u^2), \frac{\partial}{\partial u}(v^2), \frac{\partial}{\partial u}(u+2v) \right\rangle$

$$= \langle 2u, 0, 1 \rangle$$

$$\vec{r}_v(u, v) = \left\langle \frac{\partial}{\partial v}(u^2), \frac{\partial}{\partial v}(v^2), \frac{\partial}{\partial v}(u+2v) \right\rangle$$

$$= \langle 0, 2v, 2 \rangle$$

Hence a normal vector to the tangent plane is

$$\vec{n} = \vec{r}_u(1, 1) \times \vec{r}_v(1, 1)$$

$$= \langle 2, 0, 1 \rangle \times \langle 0, 2, 2 \rangle$$

$$= \langle -2, -4, 4 \rangle.$$

So the equation of the tangent plane is

$$(-2)(x-1) + (-4)(y-1) + 4(z-3) = 0$$

or

$$x + 2y - 2z + 3 = 0.$$

4. Surface Area

Suppose surface S has a parametrization:

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

$$(u,v) \in D.$$

Surface area of S is

$$A(S) = \iint_D |\vec{r_u} \times \vec{r_v}|\, dA$$

Text-Ex 11: Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution: We need to find the parametrization first. Since S is part of the paraboloid $z = x^2 + y^2$, we have

$$\vec{r}(x,y) = x\,\vec{i} + y\,\vec{j} + (x^2+y^2)\,\vec{k}, \quad (x,y) \in D.$$

The region D is determined by

$$x^2 + y^2 \le 9 \qquad \text{(b/c "under } z = 9\text{")}$$

We compute

$$\vec{r_x} = \langle 1, 0, 2x \rangle, \quad \vec{r_y} = \langle 0, 1, 2y \rangle.$$

$$\vec{r_x} \times \vec{r_y} = \langle -2x, -2y, 1 \rangle$$

$$|\vec{r_x} \times \vec{r_y}| = \sqrt{(-2x)^2 + (-2y)^2 + 1^2} = \sqrt{4x^2 + 4y^2 + 1}$$

So $\quad \text{Area}(S) = \iint_D \sqrt{4x^2+4y^2+1}\; dA$

(b/c D is described in x, y, so we need to change to polar coordinates)

$$= \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\;(r)\;dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{12}(4r^2+1)^{\frac{3}{2}}\right]_{r=0}^{r=3} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{12}(37)^{\frac{3}{2}} - \frac{1}{12}\right) d\theta$$

$$= \left[\left(\frac{1}{12}(37)^{\frac{3}{2}} - \frac{1}{12}\right)\theta\right]_0^{2\pi}$$

$$= \frac{\pi}{6}\left((37)^{\frac{3}{2}} - 1\right).$$