16.5 Curl and Divergence

1. Divergence : For $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$,

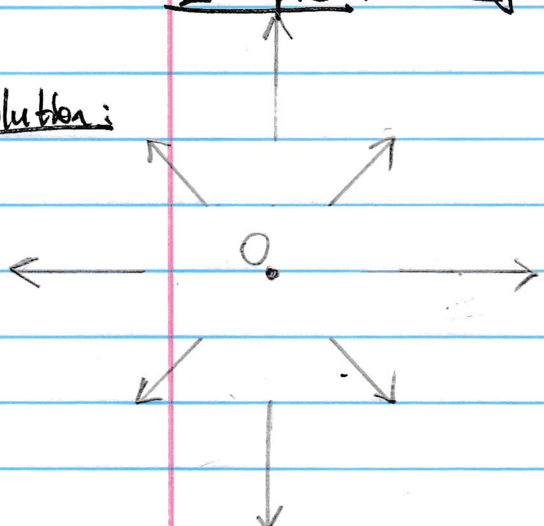
$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

↳ Two textbook notations for divergence

Divergence is a scalar and measures net gain/loss of fluid at a point (if $\vec{F}$ represents the fluid flow).

Example : If $\vec{F}(x, y, z) = x\,\vec{i} + y\,\vec{j}$. Find $\operatorname{div}\vec{F}$.

Solution:



By definition,

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(0)$$

$$= 1 + 1 + 0 = 2$$

Text-Ex 4 Find $\nabla\cdot\vec{F}$ for $\vec{F}(x, y, z) = \langle xz, xyz, -y^2\rangle$.

Solution : By definition

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$

$$= z + xz + 0 = z + xz.$$

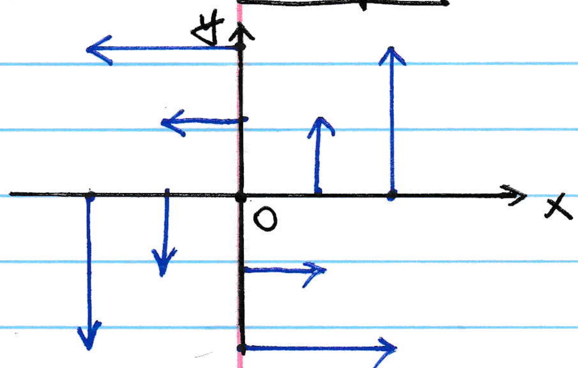
2. Curl : For $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$,

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Curl is a vector field and measures the axis of rotation of fluid at a point.

Example : If $\vec{F}(x, y, z) = -y\,\vec{i} + x\,\vec{j}$, find curl $\vec{F}$.



Solution : By definition,

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x)\right]\vec{i} + \left[\frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(0)\right]\vec{j}$$

$$+ \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right]\vec{k}$$

$$= (0)\,\vec{i} + (0)\,\vec{j} + 2\vec{k} = 2\vec{k}.$$

Text-Ex 1: For $\vec{F}(x, y, z) = \langle xz, xyz, -y^2 \rangle$, find $\nabla \times \vec{F}$.

Solution: By definition,

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz),\ \frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial x}(-y^2),\ \frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial z}(xz) \right\rangle$$

$$= \langle -2y - xy,\ x - 0,\ yz - 0 \rangle$$

$$= \langle -2y - xy,\ x,\ yz \rangle.$$

3. For $f(x, y, z)$ with continuous second-order partial derivatives,

$$\nabla \times (\nabla f) = \vec{0}.$$

For $\vec{F} = P(x, y, z)\,\vec{i} + Q(x, y, z)\,\vec{j} + R(x, y, z)\,\vec{k}$, where P, Q, R have continuous second-order partial derivatives,

$$\nabla \cdot (\nabla \times \vec{F}) = 0.$$

4. How to determine a vector field $\vec{F}$ is conservative or not.

Theorem (3D case, textbook page 1105)

If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ and $\nabla \times \vec{F} = \vec{0}$, then $\vec{F}$ is a conservative field. Also, if $\vec{F}$ is a conservative field, then $\nabla \times \vec{F} = \vec{0}$.

Text - Ex 3: Show that

$$\vec{F}(x, y, z) = y^2 z^3 \vec{i} + 2xyz^3 \vec{j} + 3xy^2z^2 \vec{k}$$

is a conservative field.

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & 3xy^2z^2 \end{vmatrix} \vec{i} + \begin{vmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ 3xy^2z^2 & y^2z^3 \end{vmatrix} \vec{j}$$

$$+ \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y^2z^3 & 2xyz^3 \end{vmatrix} \vec{k}$$

$$= \vec{0}$$

$\Rightarrow$ $\vec{F}$ is a conservative field.

Theorem (2D case, textbook pages 1090, 1091)
If $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ is a vector field defined on all of $\mathbb{R}^2$ and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

then $\vec{F}$ is a conservative field. Also, if $\vec{F}$ is a conservative field in 2D, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Text - Sec 16.3 - Ex 2: Determine whether or not the vector field $\vec{F}(x,y) = \langle \underbrace{x-y}_{=P}, \underbrace{x-2}_{=Q} \rangle$ is conservative.

Solution: $\frac{\partial P}{\partial y} = -1$, $\frac{\partial Q}{\partial x} = 1$.

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, $\vec{F}$ is not conservative.

Text - Sec 16.3 - Ex 3: Determine whether or not the vector field $\vec{F}(x,y) = (3+2xy)\,\vec{i} + (x^2-3y^2)\,\vec{j}$ is conservative.

Solution: $\frac{\partial}{\partial y}(\overset{P}{\overset{\|}{3+2xy}}) = 2x$, $\frac{\partial}{\partial x}(\overset{Q}{\overset{\|}{x^2-3y^2}}) = 2x$.

Since $\frac{\partial}{\partial y}(3+2xy) = \frac{\partial}{\partial x}(x^2-3y^2)$,

$\vec{F}$ is conservative.

5. Finding a potential function: If $\vec{F}$ is conservative, then we can use the following process to find a corresponding potential function f. Sometimes f can also be guessed directly without the process.

Example: $\vec{F}(x,y,z) = \langle 2xy, \ x^2+z, \ y+2z \rangle$.

Process: We want $\nabla f = \vec{F}$, i.e.

$$f_x(x,y,z) = 2xy, \quad f_y(x,y,z) = x^2+z, \quad f_z(x,y,z) = y+2z$$

Step 1: From $f_x(x,y,z) = 2xy$, we obtain

$$f(x,y,z) = \int 2xy \, dx + g(y,z)$$
$$= x^2 y + g(y,z) \qquad (\Delta)$$

Step 2: Plug $f(x,y,z) = x^2y + g(y,z)$ into $f_y(x,y,z) = x^2+z$ and get:

$$x^2 + z = \frac{\partial}{\partial y}\left(x^2 y + g(y,z)\right) = x^2 + g_y(y,z)$$

$$\Rightarrow \quad g_y(y,z) = z.$$

So $\quad g(y,z) = \int z \, dy + h(z) = yz + h(z)$.

Recall equation (Δ)

and hence $\quad f(x,y,z) = x^2y + yz + h(z) \qquad (\Delta\Delta)$

Step 3: Plug $f(x,y,z) = x^2y + yz + h(z)$ into $f_z(x,y,z) = y+2z$ and have

$$y + 2z = \frac{\partial}{\partial z}\left(x^2 y + yz + h(z)\right)$$

$$= 0 + y + h'(z) = y + h'(z)$$

$$\Longrightarrow \quad h'(z) = 2z$$

So $h(z) = \int 2z \, dz + C = z^2 + C$,

and hence $\longleftarrow$ Recall equation $(\Delta\Delta)$

$$f(x, y, z) = x^2 y + yz + z^2 + C.$$

Since C can be any constant we typically use $C = 0$.

Example: If $\vec{F}(x, y, z) = \langle y^2, 2xy + e^{3z}, z + 3ye^{3z} \rangle$, find a function f such that $\nabla f = \vec{F}$.

Solution: We want to f s.t. $\nabla f = \vec{F}$ i.e.

$f_x(x, y, z) = y^2$, $f_y(x, y, z) = 2xy + e^{3z}$, $f_z(x, y, z) = z + 3ye^{3z}$

Step 1: From $f_x(x, y, z) = y^2$, we obtain

$$f(x, y, z) = \int y^2 \, dx + g(y, z)$$

$$= xy^2 + g(y, z)$$

Step 2: Plug $f(x, y, z) = xy^2 + g(y, z)$ into

$f_y(x, y, z) = 2xy + e^{3z}$ and get:

$$2xy + e^{3z} = \frac{\partial}{\partial y}\left(xy^2 + g(y, z)\right) = 2xy + g_y(y, z)$$

$\Rightarrow \quad g_y(y, z) = e^{3z}$. Hence

$g(y, z) = \int e^{3z}\, dy + h(z) = y e^{3z} + h(z)$.

So $f(x, y, z) = xy^2 + ye^{3z} + h(z)$

Step 3: Plug $f(x, y, z) = xy^2 + ye^{3z} + h(z)$ into

$f_z(x, y, z) = z + 3ye^{3z}$ and get

$$z + 3ye^{3z} = \frac{\partial}{\partial z}\left(xy^2 + ye^{3z} + h(z)\right)$$

$$= 0 + 3ye^{3z} + h'(z)$$

$\Rightarrow \quad h'(z) = z$.

So $h(z) = \int z\, dz = \frac{1}{2}z^2 + C$.

Finally,

$$f(x, y, z) = xy^2 + ye^{3z} + \frac{1}{2}z^2 + C.$$

Since we only need a function, we can choose $C = 0$ and hence

$$f(x, y, z) = xy^2 + ye^{3z} + \frac{1}{2}z^2.$$