16.4 Green's Theorem

1. Theorem: If D is a region in the xy plane and C is the boundary, oriented counterclockwise, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

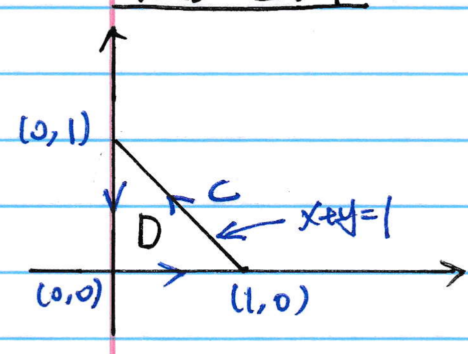
This theorem relates the line integral of a vector field to the double integral over the region contained within that curve.

Remarks:

(1) C must be closed. You sometimes see the notation $\oint_C P\,dx + Q\,dy$, which is used to indicate that the line integral is calculated on a closed curve C oriented counterclockwise.

(2) The left side $\int_C P\,dx + Q\,dy$ is the same as $\int_C (P\vec{i} + Q\vec{j}) \cdot d\vec{r}$, so keep an eye on that.

Text-Ex 1: Evaluate $\int_C x^4\,dx + xy\,dy$ where C is the triangular curve consisting of the line segments from (0,0) to (1,0), from (1,0) to (0,1), and from (0,1) to (0,0).



Solution: Apply Green's Thm:

$$\int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^4)\right) dA$$

$$= \iint_D (y - 0)\, dA$$

$$= \int_0^1 \int_0^{1-x} y \; dy \; dx$$

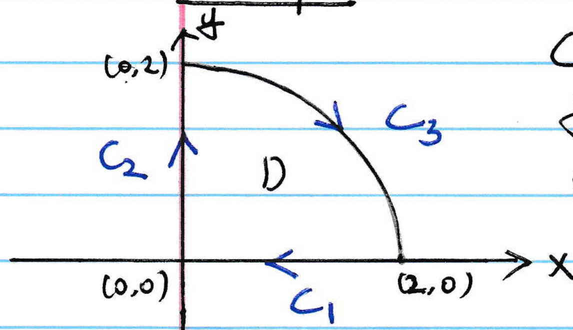
$$= \int_0^1 \left[\frac{1}{2}y^2\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left(\frac{1}{2}(1-x)^2 - \frac{1}{2}0^2\right) dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^2\, dx$$

$$= \left[-\frac{1}{6}(1-x)^3\right]_0^1 = \frac{1}{6}.$$

Example



Evaluate $\int_C xy\,dx + y\,dy$ where C consists of the line segment C_1 from $(2,0)$ to $(0,0)$, the line segment C_2 from $(0,0)$ to $(0,2)$ and the quartercircle from $(0,2)$ to $(2,0)$. ← part of $x^2+y^2=4$.

Solution: $C = C_1 \cup C_2 \cup C_3$ is oriented clockwise, we must consider $-C$ when applying Green's Thm.

$$\int_{-C} xy\,dx + y\,dy = \iint_D \left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(xy)\right) dA$$

$$= \iint_D -x \, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 -r\cos(\theta) \; r \; dr \; d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{3} r^3 \cos(\theta) \right]_{r=0}^{r=2} d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{8}{3}\cos(\theta) \; d\theta$$

$$= -\frac{8}{3}\sin(\theta) \Big|_0^{\frac{\pi}{2}} = -\frac{8}{3}$$

$$\Rightarrow \int_C xy \, dx + y \, dy$$

$$= - \int_{-C} xy \, dx + y \, dy$$

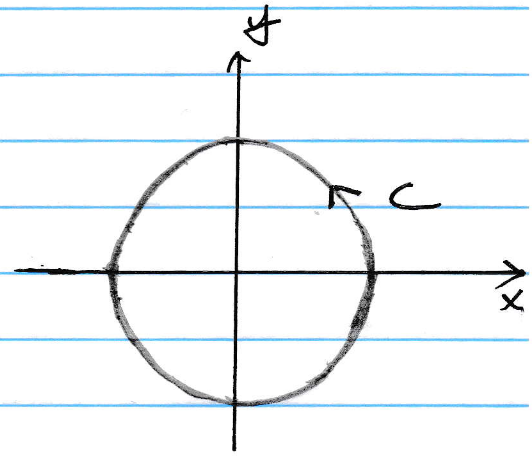
$$= -\left(-\frac{8}{3}\right) = \frac{8}{3}.$$

Example: Consider curve C:

$\vec{r}(t) = \cos(t)\,\vec{i} + \sin(t)\,\vec{j}$.

$0 \le t \le 2\pi$.

Compute $\int_C y \, dx - x \, dy$.

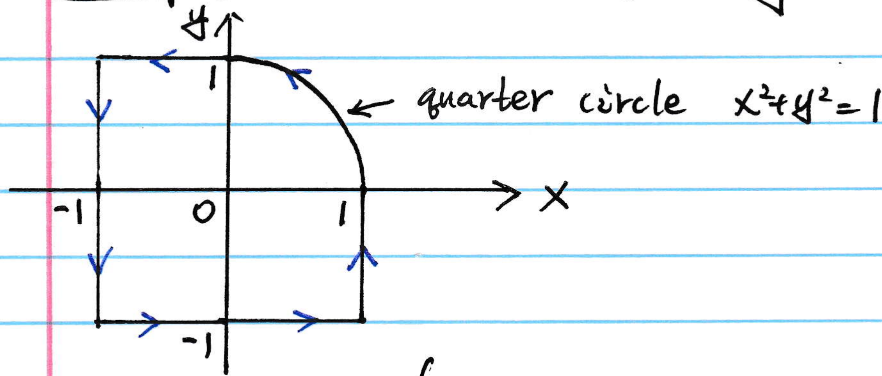


Solution: By Green's Thm,

$$\int_C y \, dx - x \, dy = \iint_D \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y) \right) dA$$

$$= \iint_D -2 \, dA = -2 \, \text{Area}(D) = -2\pi.$$

__Example__ : Consider the following curve C :



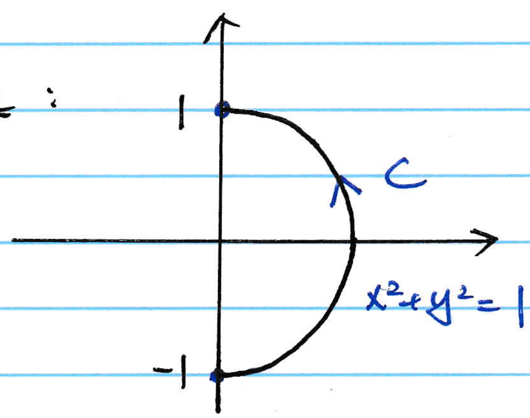
Compute $\int_C y \, dx$.

__Solution__ : By Green's Thm ,

$$\int_C y \, dx = \iint_D -\frac{\partial}{\partial y}(y) \, dA = \iint_D -1 \, dA$$

$$= -\text{Area}(D) = -\left(3 + \frac{\pi}{4}\right).$$

__Example__ :



Compute

$$\int_C y \, dx + x \, dy$$

$\left(\int_C \vec{F} \cdot d\vec{r} \right.$ with $\left. \vec{F} = \langle y, x \rangle \right)$

Notice that C is not closed, so we cannot apply Green's Thm directly. Better to use fundamental theorem of line integrals. Since for $f(x,y) = xy$, $\nabla f = \vec{F}$,

$$\int_C y \, dx + x \, dy = f(0,1) - f(0,-1) = 0.$$