16.3 The Fundamental Theorem for Line Integrals

1. Recall Fundamental Theorem of Calculus :

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

2. Fundamental Theorem of Line Integrals

$$\int_C \vec{F} \cdot d\vec{r} = f(\text{endpoint of } C) - f(\text{startpoint of } C)$$

for $\vec{F} = \nabla f$.

Remarks : (1) $\vec{F}$ must be conservative !

(2) If $\vec{F}$ is conservative and C is closed then

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

(3) The left hand side $\int_C \vec{F} \cdot d\vec{r}$ can also appear with $\int_C P\,dx + Q\,dy + R\,dz$ notation .

(4) If $\vec{F}$ is conservative, then we say the integral is independent of path because the integral only depends on the startpoint and the endpoint .

Example : $\vec{F}(x, y, z) = \langle y^2, 2xy + e^{3z}, z + 3ye^{3z} \rangle$.

Evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where C is a curve from $(0, 0, 0)$ to $(1, 1, 0)$.

Solution: From the example in the notes of section 16.5, we know $\vec{F}$ is conservative and has a potential function

$$f(x, y, z) = xy^2 + ye^{3z} + \frac{1}{2} z^2 .$$

So

$$\int_C \vec{F} \cdot d\vec{r} = f(1, 1, 0) - f(0, 0, 0)$$

$$= (1 + 1 + 0) - (0 + 0 + 0) = 2.$$

Text-Ex 4: Use Fundamental Theorem of Line Integrals to compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle \underbrace{3+2xy}_{P}, \underbrace{x^2-3y^2}_{Q} \rangle$,

and C is the curve with parametrization

$$\vec{r}(t) = e^t \sin(t) \vec{i} + e^t \cos(t) \vec{j} \qquad 0 \le t \le \pi.$$

Solution: We first check whether $\vec{F}$ is conservative.

Since $\frac{\partial Q}{\partial x} = 2x = \frac{\partial P}{\partial y}$,

we know that $\vec{F}$ is conservative.

Next we need to find f s.t. $\nabla f = \vec{F}$.

Want

$$f_x(x, y) = 3 + 2xy, \quad f_y(x, y) = x^2 - 3y^2.$$

So from $f_x(x, y) = 3 + 2xy$,

$$f(x, y) = \int (3 + 2xy)\, dx + g(y)$$

$$= 3x + x^2 y + g(y)$$

Combine $f(x,y) = 3x + x^2y + g(y)$ with $f_y(x,y) = x^2 - 3y^2$,

$$\frac{\partial}{\partial y}\left(3x + x^2 y + g(y)\right) = x^2 - 3y^2$$

$$\Rightarrow \quad 0 + x^2 + g'(y) = x^2 - 3y^2$$

$$\Rightarrow \quad g'(y) = -3y^2$$

So $\quad g(y) = \int -3y^2 \, dy = -y^3 + C.$

Therefore $\quad f(x,y) = 3x + x^2y + g(y)$

$$= 3x + x^2y - y^3 + C.$$

We only need one potential function, so we simply choose $C = 0$ and thus

$$f(x,y) = 3x + x^2y - y^3.$$

Now we apply the Fundamental Theorem of Line Integrals and have

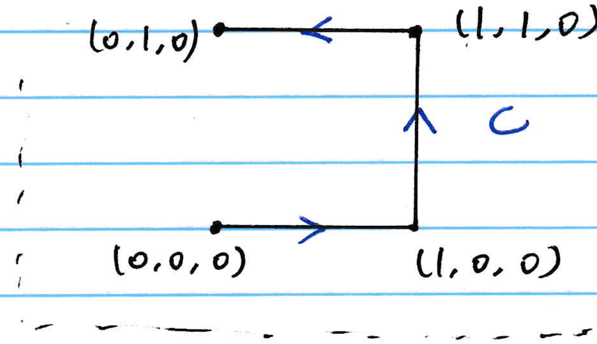
$$\int_C \vec{F} \cdot d\vec{r} = f(\underbrace{\vec{r}(\pi)}_{\uparrow \text{ endpoint of } C}) - f(\underbrace{\vec{r}(0)}_{\uparrow \text{ start point of } C})$$

$$= f(0, -e^{\pi}) - f(0, 1)$$

$$= -(-e^{\pi})^3 - \left(-(1)^3\right) = e^{3\pi} + 1.$$

Example: Compute $\int_C y^2\,dx + 2xy\,dy + z^2\,dz$

where C is the curve consisting of the line segments from $(0,0,0)$ to $(1,0,0)$, from $(1,0,0)$ to $(1,1,0)$ and from $(1,1,0)$ to $(0,1,0)$



Solution: We first find the potential function f s.t.

$$\nabla f = \vec{F} = \langle y^2, 2xy, z^2 \rangle.$$

We want $f_x = y^2$, $f_y = 2xy$, $f_z = z^2$.

From $f_x = y^2$ we have

$$f(x,y,z) = \int y^2\,dx + g(y,z) = xy^2 + g(y,z).$$

Then using $f_y = 2xy$ we get

$$\frac{\partial}{\partial y}\left(xy^2 + g(y,z)\right) = 2xy$$

$$\Rightarrow \quad 2xy + g_y(y,z) = 0$$

$$\Rightarrow \quad g_y(y,z) = 0 \quad \Rightarrow \quad g(y,z) = h(z).$$

So $f(x,y,z) = xy^2 + h(z)$.

From $f_z = z^2$ we have $\frac{\partial}{\partial z}\left(xy^2 + h(z)\right) = z^2$

$$\Rightarrow \quad 0 + h'(z) = z^2$$

$\Rightarrow \quad h'(z) = z^2$

Hence

$$h(z) = \int z^2 \, dz = \frac{1}{3} z^3 + C.$$

Choosing $C = 0$ we have $h(z) = \frac{1}{3} z^3$ and

$$f(x, y, z) = xy^2 + \frac{1}{3} z^3 .$$

So by Fundamental Theorem of Line Integrals,

$$\int_C y^2 \, dx + 2xy \, dy + z^2 \, dz$$

$$= f\underbrace{(0, 1, 0)}_{\text{endpoint of } C} - f\underbrace{(0, 0, 0)}_{\text{startpoint of } C}$$

$$= 0 - 0 = 0 .$$

<u>Example</u>: Compute $\int_C y^2 \, dx + 2xy \, dy$ where C is the curve given by the parametrization

$$\vec{r}(t) = \big(1 - \cos(t)\big)\vec{i} - \sin(t) \qquad 0 \le t \le 2\pi .$$

<u>Solution</u>: Notice that

$$\frac{\partial}{\partial x}(2xy) = 2y = \frac{\partial}{\partial y}(y^2),$$

so $\vec{F} = \langle y^2, 2xy \rangle$ is conservative.

Further notice that $\vec{r}(0) = \langle 0, 0 \rangle = \vec{r}(2\pi)$,

so C is a closed curve.

Therefore by Fundamental Theorem of Line Integrals

$$\int_C y^2 \, dx + 2xy \, dy = 0 .$$