16.2 Line Integrals

scalar-valued functions

1. Line Integrals of Functions

(1) Situation : If C is a curve representing a wire and if $f(x,y,z)$ is the density at (x,y,z) then the mass of the wire is given by the line integral ;

$$\int_C f(x,y,z)\, ds \, .$$

line integral of $f(x,y,z)$ over C

(2) Calculation : We parametrize C as

$$\vec{r}(t) = \langle x(t), \ y(t), \ z(t) \rangle = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

for $a \le t \le b$. Then

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t))\, |\vec{r}\,'(t)|\, dt$$

line integral w.r.t. arc length

Text-Ex 1 : Evaluate $\int_C (2 + x^2 y)\, ds$

where C is the upper half of the unit circle $x^2 + y^2 = 1$.

Solution : Parametrization of C :

$$x = \cos(t), \quad y = \sin(t), \qquad 0 \le t \le \pi$$

$$\int_C (2+x^2y)\,ds = \int_0^{\pi} \left(2+\cos^2(t)\sin(t)\right) |\vec{r}\,'(t)|\,dt.$$

We have $\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

$$|\vec{r}\,'(t)| = \sqrt{(-\sin(t))^2 + \cos^2(t)} = \sqrt{1} = 1.$$

So

$$\int_C (2+x^2y)\,ds.$$

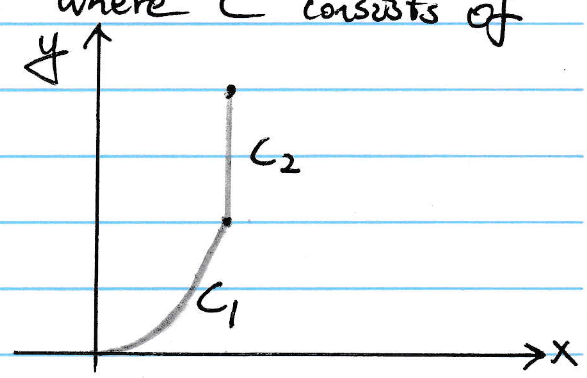
$$= \int_0^{\pi} (2+\cos^2(t)\sin(t))\,dt$$

$$= \left[2t + \left(-\frac{1}{3}\right)\cos^3(t)\right]_0^{\pi} = 2\pi + \frac{2}{3}.$$

- When C is a union of curves $C_1, \cdots, C_n$, we compute the integral over C by

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds$$

<u>Text-Ex 2</u>: Evaluate $\int_C 2x\,ds$ where C consists of the arc C_1 of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.



Solution: Parametrization of C_1 (with parameter x)

$$x = x, \quad y = x^2, \quad 0 \le x \le 1.$$

If you find this is confusing, you may rename the parameter as t:

$$x(t) = t, \quad y(t) = t^2, \quad 0 \le t \le 1.$$

Parametrization of C_2 (with parameter y)

$$x = 1, \quad y = y, \quad 1 \le y \le 2$$

We know $\int_C 2x\, ds = \int_{C_1} 2x\, ds + \int_{C_2} 2x\, ds$,

so we need to compute the integrals over C_1 and C_2 respectively. For C_1,

$$|\vec{r}\,'(x)| = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1^2 + (2x)^2} = \sqrt{1+4x^2}.$$

b/c x is the parameter

$$\int_{C_1} 2x\, ds = \int_0^1 2x\sqrt{1+4x^2}\, dx$$

$$= \left[\frac{1}{4}\cdot\frac{2}{3}(1+4x^2)^{\frac{3}{2}}\right]_0^1 = \frac{5\sqrt{5}-1}{6}$$

For C_2,

$$\int_{C_2} 2x\, ds = \int_1^2 2(1)\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\, dy$$

Here $x=1$, so $\frac{dx}{dy} = 0$, y is the parameter

$$= \int_1^2 2\, dy = 2$$

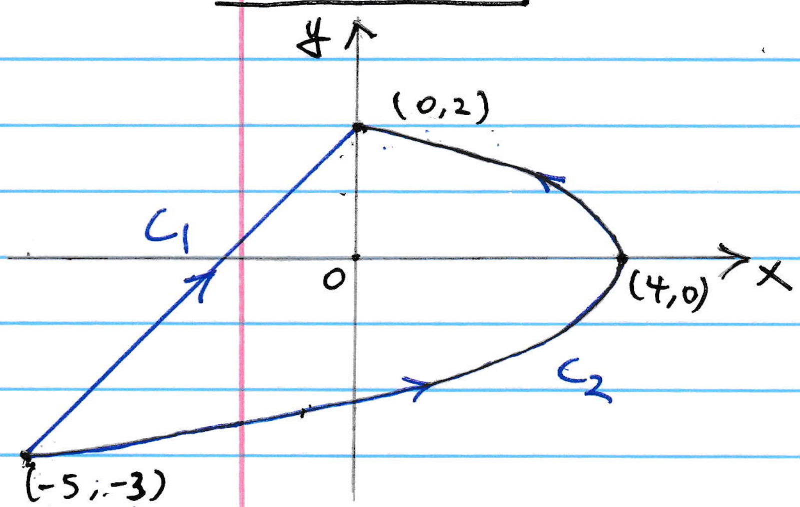
So $\int_C 2x\, ds = \int_{C_1} 2x\, ds + \int_{C_2} 2x\, ds = \boxed{\frac{5\sqrt{5}-1}{6} + 2}$

(3) Line integrals of f along C with respect to x any y.

$$\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

Text-Ex 4 Evaluate $\int_C y^2 dx + x\, dy$ where



(a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Solution (a) A parametric representation for the line segment that starts from $\vec{r_0}$ and ends at $\vec{r_1}$ is given by

$$\vec{r}(t) = \vec{r_0} + t(\vec{r_1} - \vec{r_0})$$
$$= (1-t)\vec{r_0} + t\,\vec{r_1} \qquad 0 \le t \le 1.$$

Here, a parametric representation for C_1 is:

$$\langle x(t), y(t) \rangle = \langle -5, -3 \rangle + t\left(\langle 0,2 \rangle - \langle -5,-3 \rangle\right)$$
$$= \langle -5, -3 \rangle + t\langle 5, 5 \rangle$$

$\Rightarrow \quad x(t) = -5 + 5t \quad , \quad y(t) = -3 + 5t \qquad 0 \le t \le 1.$

Then
$$\int_{C_1} y^2\,dx + x\,dy$$
$$= \int_0^1 (-3+5t)^2\, x'(t)\,dt + (-5+5t)\, y'(t)\,dt$$
$$= \int_0^1 (-3+5t)^2 (5)\,dt + (-5+5t)(5)\,dt$$
$$= \int_0^1 (-3+5t)^2 (5) + (-5+5t)(5)\,dt$$
$$= \int_0^1 125t^2 - 125t + 20\,dt$$
$$= \left[125\frac{t^3}{3} - 125\frac{t^2}{2} + 20t\right]_0^1 = -\frac{5}{6}.$$

(b) Since the parabola is given as a function of y ($x = 4 - y^2$), we could simply take y as a parameter.

$$x = 4 - y^2 \quad , \quad y = y \qquad -3 \le y \le 2.$$

Then $\dfrac{dx}{dy} = -2y \quad , \quad \dfrac{dy}{dy} = 1$

like $x'(t)$ in the formula of $\int_C f(x,y)\,dx$.

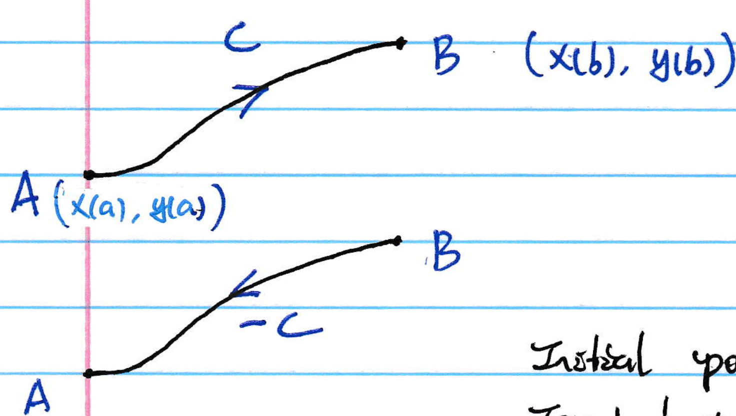
Hence
$$\int_{C_2} y^2\,dx + x\,dy$$
$$= \int_{-3}^{2} y^2(-2y)\,dy + (4-y^2)\,dy$$

$$= \int_{-3}^{2} y^2(-2y) + (4 - y^2)\, dy$$

$$= \int_{-3}^{2} -2y^3 - y^2 + 4\, dy$$

$$= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y \right]_{-3}^{2} = \frac{245}{6}.$$

(14) Orientation of a curve C



A given parametrization $x = x(t)$, $y = y(t)$, $a \leq t \leq b$. determines the orientation of a curve C

Initial point A corresponds to $t = a$.
Terminal point B corresponds to $t = b$.

$-C$ denotes the curve with opposite orientation, then we have

$$\int_{-C} f(x, y)\, dx = -\int_C f(x, y)\, dx$$

$$\int_{-C} f(x, y)\, dy = -\int_C f(x, y)\, dy$$

$$\int_{-C} f(x, y)\, ds = \int_C f(x, y)\, ds$$

If we are integrating w.r.t. arc length, then we get the same answer for C and $-C$.

2. Line Integrals of Vector Fields

(1) Situation : If C is a curve representing the path of an object through a vector field (force field) $\vec{F}(x, y, z) = P\vec{i} + Q\vec{j} + R\vec{k}$,

then the work done by $\vec{F}$ on the object is given by the line integral

$$\int_C \vec{F} \cdot d\vec{r} \quad \text{or} \quad \int_C P\,dx + Q\,dy + R\,dz$$

(2) Calculation : We parametrize C as

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \le t \le b$,

then

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\,dt.$$

<u>Remarks</u> : Orientation of C matters.

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$

<u>Text-Ex 1</u> Find the work done by the force field $\vec{F}(x, y) = x^2\,\vec{i} - xy\,\vec{j}$ in moving a particle along the quarter-circle $\vec{r}(t) = \cos(t)\,\vec{i} + \sin(t)\,\vec{j}$

$0 \le t \le \frac{\pi}{2}$.

vector field $\vec{F}$ here

Solution: We first write the integrand function in terms of the parameter t

$$\vec{F}(\vec{r}(t)) = \langle \cos^2(t) \, , \, -\cos(t)\sin(t) \rangle .$$

Also, $\vec{r}\,'(t) = \langle -\sin(t) \, , \, \cos(t) \rangle .$

Hence

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$= \langle \cos^2(t) \, , \, -\cos(t)\sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle$$

$$= \cos^2(t)\,(-\sin(t)) + (-\cos(t)\sin(t))\cos(t)$$

$$= -2\cos^2(t)\sin(t) .$$

Finally,

$$\text{Work} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{\frac{\pi}{2}} -2\cos^2(t)\sin(t)\, dt$$

$$= \left[\frac{2}{3}\cos^3(t) \right]_0^{\frac{\pi}{2}} = -\frac{2}{3} .$$

Text-Ex 8 Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where

$\vec{F}(x, y, z) = xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{k}$ and C is the curve with parametrization

$$x = t, \quad y = t^2, \quad z = t^3 \qquad 0 \le t \le 1 .$$

Solution: Write $\vec{F}$ using t:

$$\vec{F}(\vec{r}(t)) = t^3\,\vec{i} + t^5\,\vec{j} + t^4\,\vec{k}.$$

Also, $\vec{r}\,'(t) = (1)\,\vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k}$.

So
$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = t^3(1) + t^5(2t) + t^4(3t^2)$$
$$= t^3 + 2t^6 + 3t^6$$
$$= 5t^6 + t^3$$

and
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$
$$= \int_0^1 (5t^6 + t^3)\,dt = \left[\frac{5}{7}t^7 + \frac{1}{4}t^4\right]_0^1 = \frac{27}{28}.$$

3. Comments on the notations

In this section, we have discussed notations like

$$\int_C x\,dx, \quad \int_C x\,ds.$$

They have different meanings.

- Q: Which of the following integrals make sense and which do not?

(a) $\int_C x\,dx$ Makes sense. Line integral w.r.t x.

(b) $\int_C x\,dy$ Makes sense. Line integral w.r.t. y.

(c) $\int_0^1 x\,dx$ Makes sense. 1D integral.

(d) $\int_0^1 x\,ds$ No sense.

(e) $\int_C x\,ds$ Makes sense. Line integral w.r.t arc length.

(f) $\int_0^1 s\,ds$ Makes sense. 1D integral.