16.1 Vector Fields

1. Definition : A vector field is a function which assigns a vector to each point. This can either be 2D or 3D.

Examples :

(1) $\vec{F}(x, y) = \langle y, \sin(x) \rangle = y\,\vec{i} + \sin(x)\,\vec{j}$

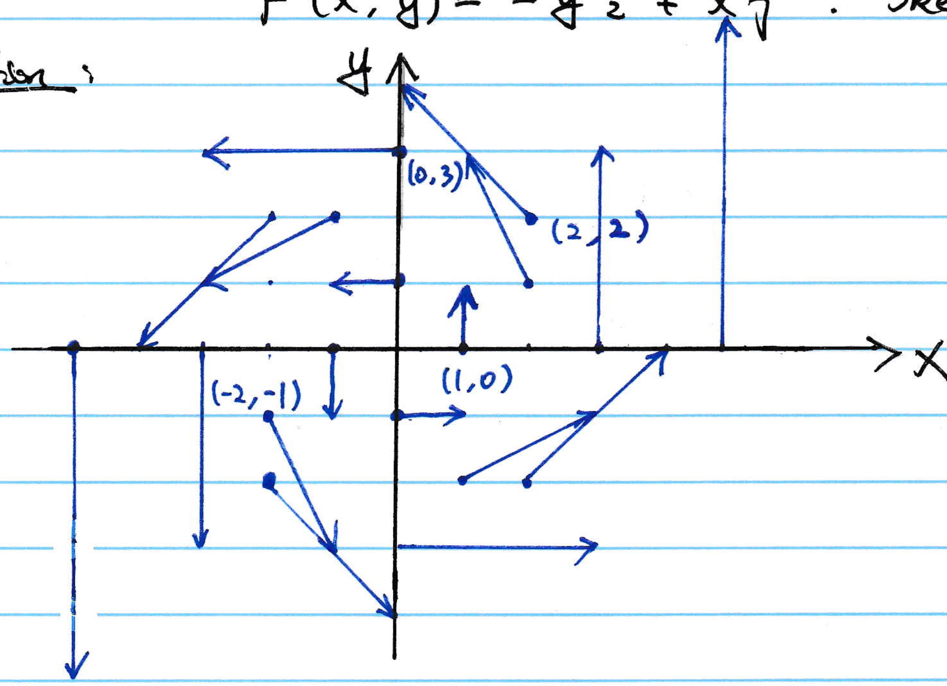
(2) $\vec{F}(x, y, z) = \langle y, -2, x \rangle = y\,\vec{i} - 2\,\vec{j} + x\,\vec{k}$

(3) $\vec{F}(x, y, z) = \langle 0, 0, z \rangle = z\,\vec{k}$

2. We can sketch a vector field by simply plotting a vector at each point for a nice selection of points.

Text-Example 1 : A vector field on $\mathbb{R}^2$ is defined by $\vec{F}(x, y) = -y\,\vec{i} + x\,\vec{j}$. Sketch $\vec{F}$.

Solution :



More Examples in the textbook :

Text-Ex 2 : $\vec{F}(x, y, z) = z\,\vec{k}$

Text-Ex 4 : Gravitational field

$$\vec{F}(x, y, z) = \left\langle \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMGy}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

Textbook Figures : 7, 8, 11, 12, 13.

- Vector fields can be thought as any sort of force fields, electric fields, fluid flow, etc.

3. Gradient Fields

We have seen vector fields before. Recall that the gradient

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

which gives us a vector field. This is called a gradient vector field.

Text-Ex 6 : Find the gradient vector field

$$f(x, y) = x^2 y - y^3 .$$

Solution : $\nabla f(x, y) = \langle 2xy, (x^2 - 3y^2) \rangle$.

See figure 15 in the textbook for the plot of ∇f and the contour map of f. We see that gradient vectors are perpendicula to the level curves, and the gradient vetors are long where the level curves are close to each other.

Definition : A vector field $\vec{F}$ is conservative if there is some function f so that $\vec{F} = \nabla f$. This function f is called a potential function.

Example : $\vec{F}(x, y, z) = y^2 z \vec{i} + 2xyz \vec{j} + xy^2 \vec{k}$ is conservative because

$$\vec{F}(x, y, z) = \nabla f(x, y, z)$$

for $f(x, y, z) = xy^2 z + C$.

Notice that the potential function is determined up to a constant. We will discuss how to find $f(x, y, z)$ from $\vec{F}(x, y, z)$ in section 16.3.