15.9 Change of Variables in Multiple Integrals

1. Motivation: For double integrals, sometimes vertically simple, horizontally simple and polar coordinates are insufficient for parallelograms, ellipse and other quirky shapes

2. Recall change of variables for integrals in 1D.

For $\int_0^1 \sqrt{1-x^2}\,dx$, we let $x = \sin(u)$ and get

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2(u)} \quad \cos(u)\,du \;.$$

Three things have changed: the interval, the integrad function, and the dx is replaced by $\cos(u)du$.

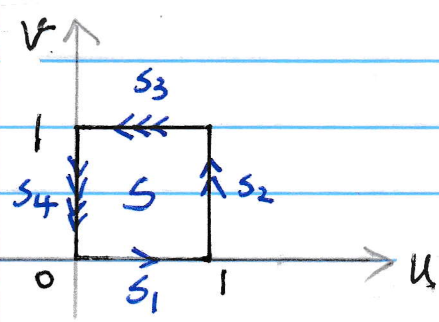
3. Transformation in 2D: An Example

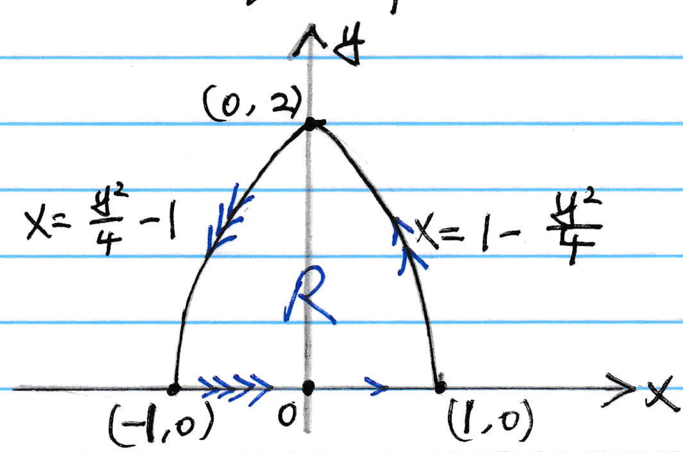
Text-Ex 1: A transformation is defined by the equations

$$x = u^2 - v^2, \quad y = 2uv.$$

Find the image of the square $S = \{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$.

Solution:





Fos S_1 : $v = 0$, $0 \le u \le 1$.
It is mapped to $x = u^2$, $y = 0$, $0 \le u \le 1$.
The image is $0 \le x \le 1$, $y = 0$.

For S_2 : $u = 1$, $0 \le v \le 1$
Mapped to $x = 1 - v^2$, $y = 2v$ (so $v = \frac{y}{2}$)
The image of S_2 is $x = 1 - (\frac{y}{2})^2$, $0 \le y \le 2$.

For S_3 : $v = 1$, $0 \le u \le 1$
Mapped to $x = u^2 - 1$, $y = 2u$ (so $u = \frac{y}{2}$)
The image of S_3 is $x = (\frac{y}{2})^2 - 1$, $0 \le y \le 2$.

For S_4 : $u = 0$, $0 \le v \le 1$
Mapped to $x = -v^2$, $y = 0$
The image is $-1 \le x \le 0$, $y = 0$.
Combine the above images of S_1, S_2, S_3, S_4 , we can plot the image of region S on xy-plane which is denoted by R.

• <u>Definition of Jacobian of a transformation</u>

$$\underbrace{\frac{\partial(x,y)}{\partial(u,v)} = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right)}_{J_{(x,y)}(u,v)\text{ , Jacobian}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

For Text-Ex 1, the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$$= (2u)(2u) - (-2v)(2v) = 4u^2 + 4v^2.$$

4. Change of variables for a double integral

If R is a not-so-nice region and we want to evaluate $\iint_R f(x,y)\,dA$ and if we can do a transformation (substitution) $x = g(u,v)$, $y = h(u,v)$ as functions of u, v which changes R (in the xy-plane) into S (in the uv-plane) then:

S should be a nice region

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA$$

absolute value of Jacobian.

Remarks: (1) Often the change of variables are given by $u =$ and $v =$. So we need to solve for $x =$ and $y =$ if we need them for the integrand.

(2) Sometimes it might be helpful to use

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

in the computation of the Jacobian.

Text-Ex 2: Use change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y \, dA$

where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

Solution: From Text-Ex 1, region S is (on uv-plane) described as $0 \leq u \leq 1$, $0 \leq v \leq 1$.

So

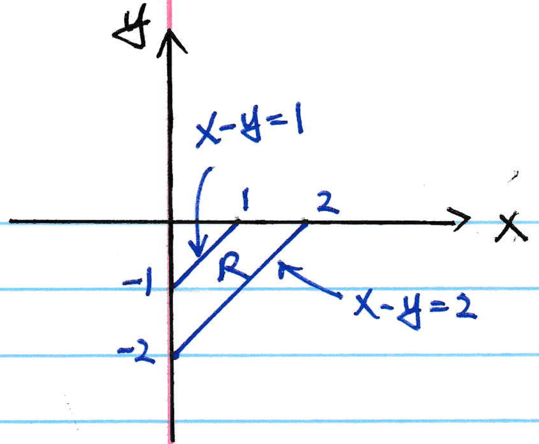
$$\iint_R y \, dA = \iint_S 2uv \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA$$

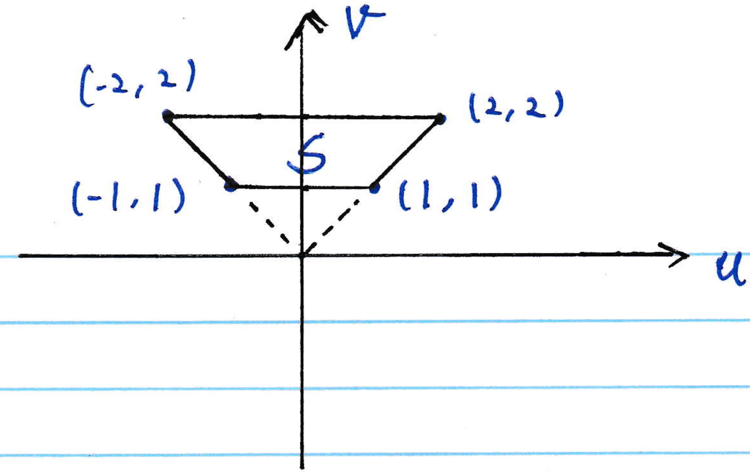
$$= \int_0^1 \int_0^1 2uv \, (4u^2 + 4v^2) \, du \, dv$$

Text-Ex 3: Evaluate the integral $\iint_R e^{\frac{(x+y)}{(x-y)}} \, dA$

where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$.

(If in the exam, transformation $u = x + y$, $v = x - y$ will be given)





<u>Solution</u>: From the shape of R in the xy-plane, it may be natural to have a transformation with

$$v = x - y.$$

What about u? In fact if we only want to transform the integral over R into an integral over a nice region, we have many choices. However, there is one choice that makes the integrand function simpler at the same time.

$$u = x + y, \qquad v = x - y.$$

Then the integrand becomes $e^{u/v}$.

For $\dfrac{\partial(x,y)}{\partial(u,v)}$, we use the formula

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}},$$

and compute

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= (1)(-1) - (1)(1) = -2.$$

So $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{-2} = -\dfrac{1}{2}$.

An alternative method of computing $\dfrac{\partial(x,y)}{\partial(u,v)}$ is discussed in the textbook. You first solve x, y from

$$\begin{cases} u = x + y \\ v = x - y \end{cases}$$

and obtain expressions of x, y in terms of u, v:

$$x = \frac{1}{2}(u+v), \qquad y = \frac{1}{2}(u, v).$$

Then:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}.$$

Now to find the region S in the uv-plane, we notice that the sides of R are given by equations:

$$y = 0 \quad ; \quad x - y = 2 \quad ; \quad x = 0 \quad ; \quad x - y = 1$$

respectively. These become

$$u = v \quad ; \quad v = 2 \quad ; \quad u = -v \quad ; \quad v = 1$$

in terms of u and v.

So S is the trapezoidal region with vertices $(1,1)$, $(2,2)$, $(-2,2)$ and $(-1,1)$. See plots at the beginning of the solution or the ones in the book.

Hence S is described as (horizontally simple)

$$1 \le v \le 2 \quad , \quad -v \le u \le v .$$

By the formula of change of variables.

$$\iint_R e^{\frac{x+y}{x-y}}\, dA = \iint_S e^{\frac{u}{v}} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA$$

$$= \iint_S e^{\frac{u}{v}} \left| -\frac{1}{2} \right| dA$$

$$= \int_1^2 \int_{-v}^{v} \frac{1}{2} e^{\frac{u}{v}}\, du\, dv$$

$$= \int_1^2 \left[\frac{1}{2} v\, e^{\frac{u}{v}} \right]_{u=-v}^{u=v} dv$$

$$= \int_1^2 \left(\frac{1}{2} v e - \frac{1}{2} v e^{-1} \right) dv$$

$$= \left(\frac{1}{2} e - \frac{1}{2} e^{-1} \right) \int_1^2 v\, dv = \frac{3}{4} (e - e^{-1}) .$$

<u>Example</u>: Find $\iint_R y^2\, dA$ where R is the region inside $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Use change of variables $u = \frac{x}{3}$, $v = \frac{y}{2}$.

Solution: Rewrite $\frac{x^2}{9} + \frac{y^2}{4} = 1$ as:

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1.$$

So after change of variables, the equation becomes

$$u^2 + v^2 = 1.$$

Then S is the unit disk (inside $u^2+v^2=1$).

Since $x = 3u$, $y = 2v$ from the change of variables, we have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}$$

$$= 6$$

$$\iint_R y^2\, dA = \iint_S (2v)^2\,(6)\, dA$$

Polar coordinates
$u = r\cos(\theta)$,
$v = r\sin(\theta)$

$$= \int_0^{2\pi}\int_0^1 \left(2r\sin(\theta)\right)^2 6r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 24r^3\sin^2(\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[6r^4\sin^2(\theta)\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} 6\sin^2(\theta)\,d\theta$$

$$= \int_0^{2\pi} 3\left(1-\cos(2\theta)\right) d\theta$$

$$= \left[3\theta - \frac{3}{2}\sin(2\theta)\right]_0^{2\pi} = 6\pi.$$

Example: Use change of variables and set up an iterated integral for $\iint_R y \, dA$,

where R is the region bounded by $y = 2x-1$, $y = 2x+1$, $y = 1-x$, $y = 3-x$.

Solution: You need to come up with the change of variables by yourself for this kind of problem.

Rewrite the equations for the boundary of R:

$$y = 2x-1 \longrightarrow y-2x = -1$$

$$y = 2x+1 \longrightarrow y-2x = 1$$

$$y = 1-x \longrightarrow y+x = 1$$

$$y = 3-x \longrightarrow y+x = 3.$$

Let $u = x+y$, $v = y-2x$, then the equations for the boundary become

$$v = -1, \quad v = 1, \quad u = 1, \quad u = 3.$$

So the region S is a rectangle:

$$1 \le u \le 3, \quad -1 \le v \le 1.$$

Solve x, y in terms of u, v from

$$\begin{cases} u = x+y \\ v = y-2x \end{cases}$$

and obtain

$$x = \frac{u-v}{3}, \quad y = \frac{2u+v}{3}.$$

Here $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) - \left(-\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{1}{3}.$$

So $\iint_R y \, dA = \iint_S \frac{2u+v}{3} \left|\frac{1}{3}\right| dA$

$$= \int_1^3 \int_{-1}^1 \frac{2u+v}{9} \, dv \, du.$$

Example Use change of variables and set up an iterated integral for $\iint_R xy \, dA$,

where R is bounded by $y = x$, $y = 2x$, $y = \frac{1}{x}$, $y = \frac{2}{x}$ in the first quadrant.

Solution: Rewrite the equations for the boundary of R as:

$$\frac{y}{x} = 1, \quad \frac{y}{x} = 2, \quad xy = 1, \quad xy = 2.$$

Let $u = xy$ and $v = \frac{y}{x}$, then the equations become

$$v = 1, \quad v = 2, \quad u = 1, \quad u = 2.$$

So S is: $1 \le u \le 2$, $1 \le v \le 2$.

From $u = xy$ and $v = \frac{y}{x}$, we can solve x, y in terms of u, v. To be specific,

$$uv = (xy)\left(\frac{y}{x}\right) = y^2 .$$

$$\frac{u}{v} = \frac{xy}{\frac{y}{x}} = x^2 .$$

Since $x, y > 0$ (b/c first quadrant), we have

$$x = \sqrt{\frac{u}{v}} , \quad y = \sqrt{uv} .$$

So $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix}$$

$$= \left(\frac{1}{2\sqrt{uv}}\right)\left(\frac{\sqrt{u}}{2\sqrt{v}}\right) - \left(-\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}}\right)\left(\frac{\sqrt{v}}{2\sqrt{u}}\right)$$

$$= \frac{1}{4v} + \frac{1}{4v} = \frac{1}{2v}$$

Hence $$\iint_R xy \, dA = \iint_S u \left|\frac{1}{2v}\right| dA$$

$$= \int_1^2 \int_1^2 \frac{u}{2v} \, dv \, du .$$

5. Triple Integrals (Not Required)

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

Text - Ex 4: Derive the formula for triple integration in spherical coordinates.

Solution: We let (ρ, θ, ϕ) be (u, v, w) in the formula above. Since

$$x = \rho \sin(\phi) \cos(\theta), \quad y = \rho \sin(\phi) \sin(\theta), \quad z = \rho \cos(\phi).$$

we compute the Jacobian as follows:

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin(\phi)\cos(\theta) & -\rho \sin(\phi) \sin(\theta) & \rho \cos(\phi) \cos(\theta) \\ \sin(\phi) \sin(\theta) & \rho \sin(\phi) \cos(\theta) & \rho \cos(\phi) \sin(\theta) \\ \cos(\phi) & 0 & -\rho \sin(\phi) \end{vmatrix}$$

$$= \sin(\phi) \cos(\theta) \begin{vmatrix} \rho \sin(\phi) \cos(\theta) & \rho \cos(\phi) \sin(\theta) \\ 0 & -\rho \sin(\phi) \end{vmatrix}$$

$$- \left(-\rho \sin(\phi)\sin(\theta)\right) \begin{vmatrix} \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) \\ \cos(\phi) & -\rho\sin(\phi) \end{vmatrix}$$

$$+ \rho\cos(\phi)\cos(\theta) \begin{vmatrix} \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \\ \cos(\phi) & 0 \end{vmatrix}$$

$$= \sin(\phi)\cos(\theta) \;\; \rho\sin(\phi)\cos(\theta) \;\; \left(-\rho\sin(\phi)\right)$$

$$+ \rho\sin(\phi)\sin(\theta)\left(-\rho\sin^2(\phi)\sin(\theta) - \rho\cos^2(\phi)\sin(\theta)\right)$$

$$+ \rho\cos(\phi)\cos(\theta)\left(-\rho\sin(\phi)\cos(\phi)\cos(\theta)\right)$$

$$= -\rho^2\sin^3(\phi)\cos^2(\theta) - \rho^2\sin(\phi)\sin^2(\theta)$$

$$- \rho^2\sin(\phi)\cos^2(\phi)\cos^2(\theta)$$

$$= -\rho^2\sin(\phi)\cos^2(\theta) - \rho^2\sin(\phi)\sin^2(\theta)$$

$$= -\rho^2\sin(\phi).$$

So $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|-\rho^2\sin(\phi)\right| = \rho^2\sin(\phi)$

$$\iiint_R f(x,y,z)\,dV = \iiint_S f\left(\rho\sin(\phi)\cos(\theta),\ \rho\sin(\phi)\sin(\theta),\ \rho\cos(\phi)\right)$$

$$\underline{\rho^2\sin(\phi)} \quad d\rho\, d\theta\, d\phi$$

We get the additional factor for spherical