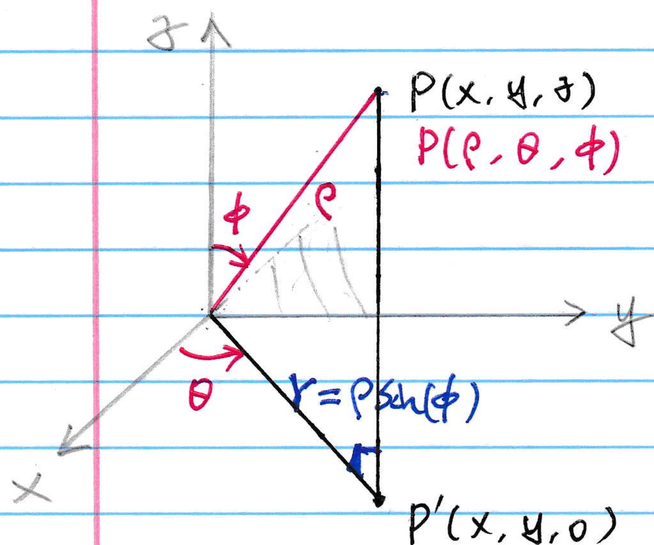
15.8 Triple Integrals in Spherical Coordinates

1. Introduction : In spherical coordinates, we locate a point using three variables :

- θ is the familiar one, the angle from the positive x-axis. $(0 \le \theta \le 2\pi)$
- ϕ is the angle measured from the positive z-axis downwards, so $0 \le \phi \le \pi$.
- ρ is the distance from the origin. $(\rho \ge 0)$



$$\boxed{\begin{aligned} x &= \rho \sin(\phi) \cos(\theta) \\ y &= \rho \sin(\phi) \sin(\theta) \\ z &= \rho \cos(\phi) \end{aligned}}$$

Convert from spherical to rectangular coordinates.

Convert from rectangular to spherical coordinates:

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\cos(\phi) = \frac{z}{\rho} \Rightarrow \phi = \cos^{-1}\left(\frac{z}{\rho}\right)$$

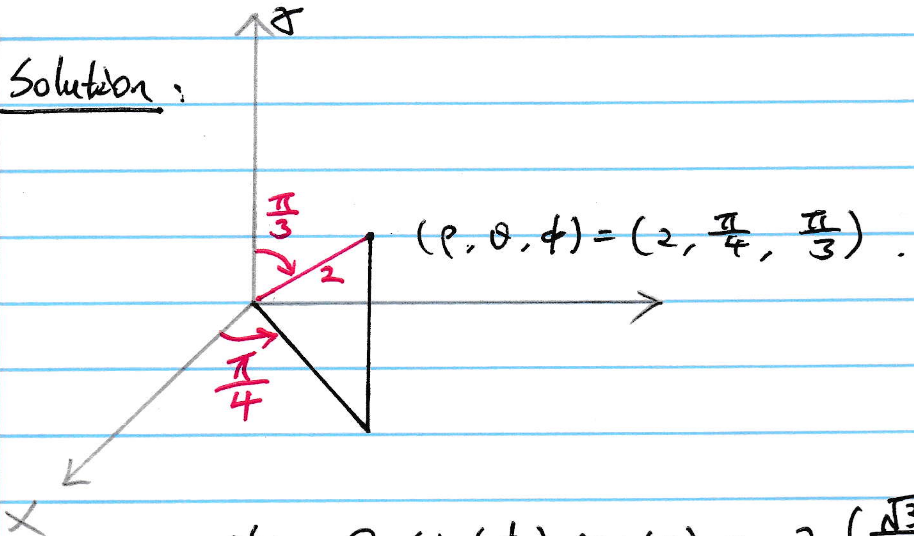
Use $\cos(\theta) = \dfrac{x}{\rho \sin(\phi)}$, sign of y

to determine θ.

(ρ, θ, ϕ)

Text-Ex 1: The point $(2, \frac{\pi}{4}, \frac{\pi}{3})$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

Solution:



$$x = \rho \sin(\phi)\cos(\theta) = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{2}$$

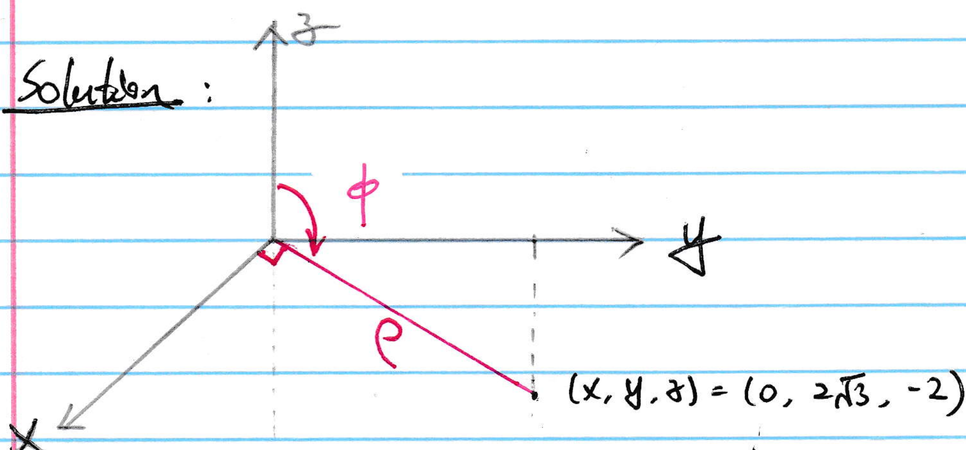
$$y = \rho \sin(\phi)\sin(\theta) = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{2}$$

$$z = \rho\cos(\phi) = 2\left(\frac{1}{2}\right) = 1.$$

(x, y, z)

Text-Ex The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates of this point.

Solution:



$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0^2 + (2\sqrt{3})^2 + 2^2} = 4$$

$$\cos(\phi) = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \implies \phi = \frac{2\pi}{3}$$

(Note: $\phi \in [0, \pi]$)

For θ, notice that $y = 2\sqrt{3} > 0$, so $0 \leq \theta \leq \pi$.

Also,
$$\cos(\theta) = \frac{x}{\rho \sin(\phi)} = 0,$$

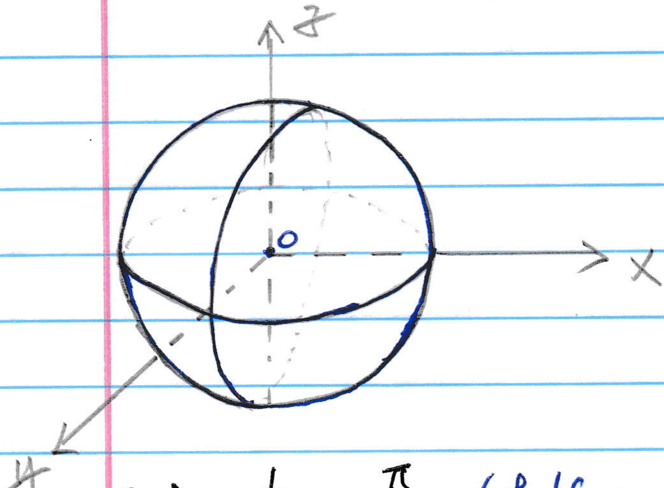
so $\theta = \frac{\pi}{2}$.

Therefore spherical coordinates $(\rho, \theta, \phi) = (4, \frac{\pi}{2}, \frac{2\pi}{3})$.

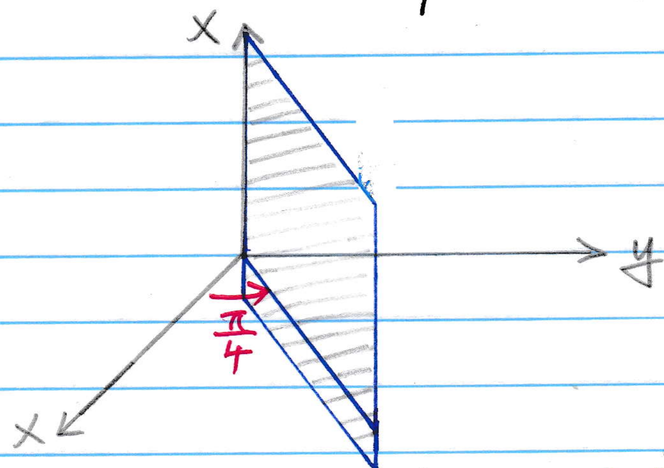
2. Equations of surfaces in spherical coordinates.

Examples:

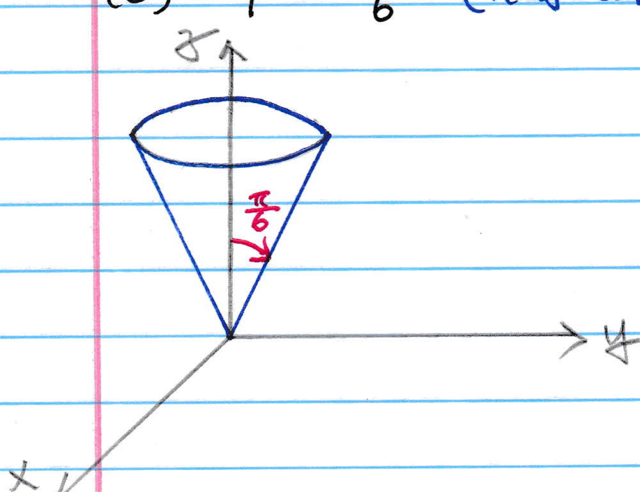
(a) $\rho = 2$ (sphere)



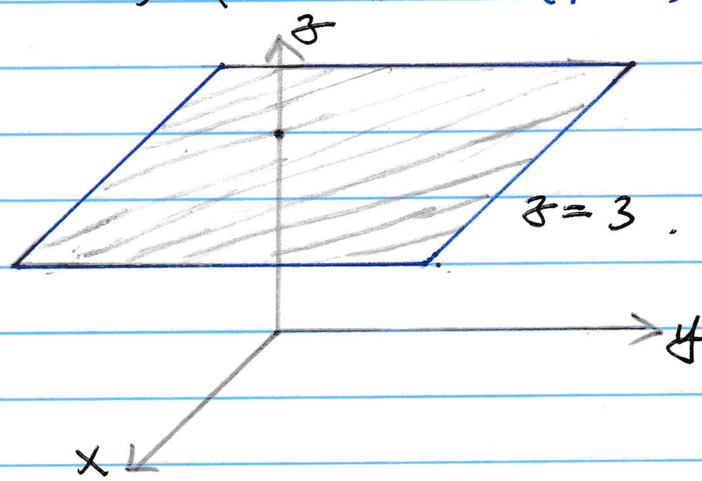
(b) $\theta = \frac{\pi}{4}$ (half-plane)



(c) $\phi = \frac{\pi}{6}$ (half-cone)

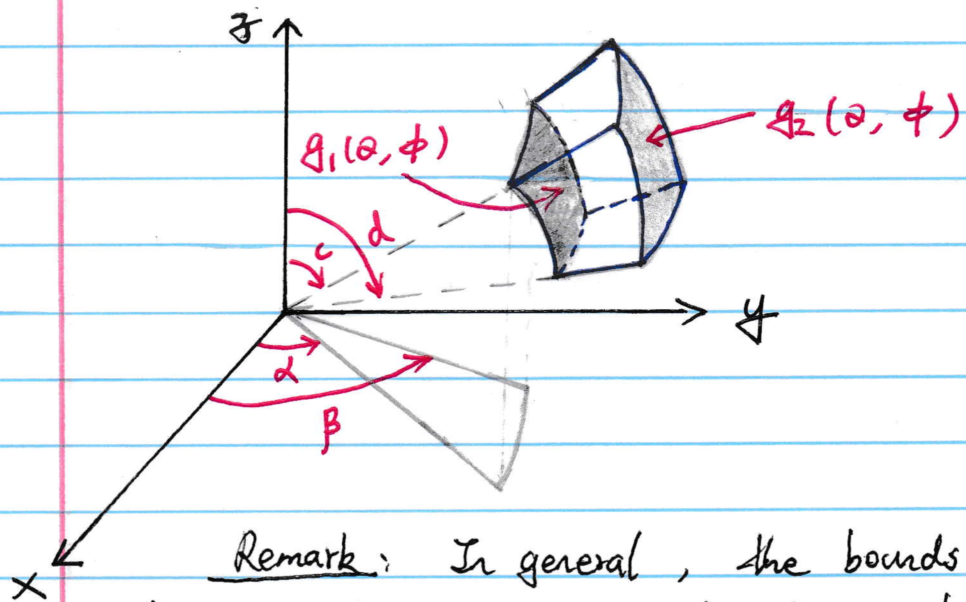


(d) $\rho \cos(\phi) = 3$ (plane)



3. Describing solids in spherical:

$\alpha \leq \theta \leq \beta$, $c \leq \phi \leq d$, $g_1(\theta,\phi) \leq \rho \leq g_2(\theta,\phi)$



Remark: In general, the bounds for ϕ can be $h_1(\theta)$ and $h_2(\theta)$: $h_1(\theta) \leq \phi \leq h_2(\theta)$. However, we will only consider the case where we have constants bounds for ϕ.

4. $$\iiint_E f(x,y,z)\, dV$$

$$= \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\, \rho^2\sin(\phi)\, d\rho\, d\theta\, d\phi$$

Remark: The $\rho^2\sin(\phi)$ is like the polar/cylindrical r.

Example: Consider the solid E between the spheres $\rho = 1$ and $\rho = 2$. Write the triple integral $\iiint_E 1 \, dV$ as an iterated integral using spherical coordinates.

Solution: E is described as

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi, \quad 1 \leq \rho \leq 2.$$

So $$\iiint_E 1 \, dV = \int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta.$$

Example: Consider the previous example but let E be the solid region between $\rho = 1$ and $\rho = 2$, and satisfying $x \leq 0$, $y \geq 0$, $z \geq 0$.

Solution: $z \geq 0$ in spherical is $\rho \cos(\phi) \geq 0$.

So $0 \leq \phi \leq \frac{\pi}{2}$.

Since $$x = \underbrace{\rho \sin(\phi)}_{r} \cos(\theta) = r \cos(\theta) \leq 0$$

$$y = \rho \sin(\phi) \sin(\theta) = r \sin(\theta) \geq 0,$$

we have $$\frac{\pi}{2} \leq \theta \leq \pi.$$

Therefore

$$\iiint_E 1 \, dV = \int_{\frac{\pi}{2}}^{\pi} \int_0^{\frac{\pi}{2}} \int_1^2 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta.$$

Text-Ex 3: Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$ where B is the unit ball $B = \{(x, y, z) \mid x^2+y^2+z^2 \le 1\}$.

Solution: In spherical coordinates, B is

$$\underbrace{0 \le \rho \le 1}, \quad \underbrace{0 \le \theta \le 2\pi}, \quad \underbrace{0 \le \phi \le \pi}.$$

b/c $\rho^2 = x^2+y^2+z^2 \le 1$ and it is always true $\rho \ge 0$.

We always have $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$ in spherical coordinates.

The integrand function $e^{(x^2+y^2+z^2)^{\frac{3}{2}}}$ is

$$e^{(\rho^2)^{\frac{3}{2}}} = e^{\rho^3}$$ in spherical coordinates.

So $\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{\rho^3} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[\frac{1}{3} e^{\rho^3} \sin(\phi)\right]_{\rho=0}^{\rho=1} d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{e-1}{3} \sin(\phi)\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{e-1}{3}\cos(\phi)\right]_{\phi=0}^{\phi=\pi} d\theta$$

$$= \int_0^{2\pi} \frac{2(e-1)}{3}\, d\theta = \frac{4\pi(e-1)}{3}.$$

Text-Ex 4 : Find the volume of the solid E that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$. (See Figure 9 in the textbook)

Solution : First write the equations $z = \sqrt{x^2+y^2}$ and $x^2+y^2+z^2 = z$ using (ρ, θ, ϕ).

For $z = \sqrt{x^2+y^2}$, we have

$$\rho\cos(\phi) = \sqrt{(\rho\sin(\phi)\cos(\theta))^2 + (\rho\sin(\phi)\sin(\theta))^2}$$

$$= \rho\sin(\phi)$$

$$\Rightarrow \quad 1 = \tan(\phi) \quad \Rightarrow \quad \phi = \frac{\pi}{4}.$$

For $x^2+y^2+z^2 = z$, we have

$$\rho^2 = \rho\cos(\phi) \quad \Rightarrow \quad \rho = 0 \text{ or } \rho = \cos(\phi).$$

Since $\rho = 0$ is included in $\rho = \cos(\phi)$ (when $\phi = \frac{\pi}{2}$, $\cos(\phi) = 0$), so $x^2+y^2+z^2 = z$ is $\rho = \cos(\phi)$ in spherical coordinates.

E is described as : From the derivations above

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{4}, \quad 0 \le \rho \le \cos(\phi)$$

Therefore Volume $= \iiint_E 1 \, dV$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos(\phi)} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta = \cdots = \frac{\pi}{8}.$$