15.7 Triple Integrals in Cylindrical Coordinates

1. Introduction: Cylindrical is like polar plus z.

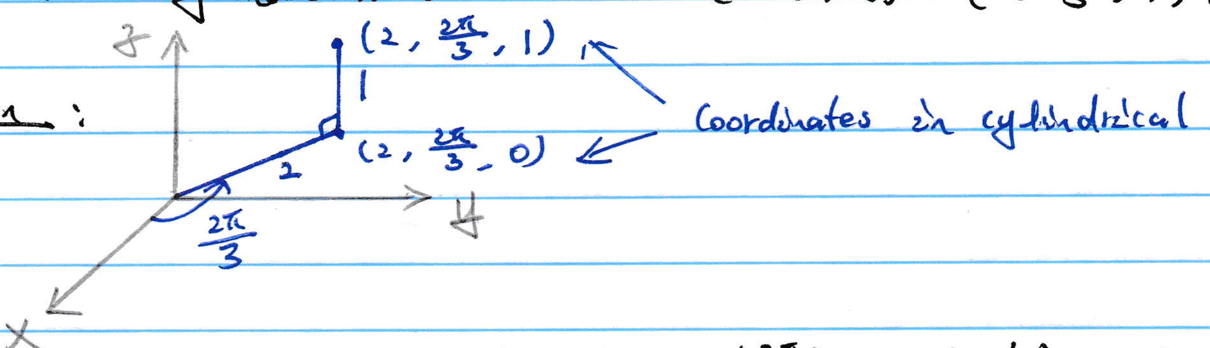
$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z.$$

The above tells us how to convert from cylindrical to rectangular, whereas to convert from rectangular to cylindrical:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z.$$

Text-Ex 1: (a) Find the rectangular coordinates for a point with cylindrical coordinates $(r, \theta, z) = (2, \frac{2\pi}{3}, 1)$.

Solution:



$$x = r\cos(\theta) = 2\cos\left(\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2}\right) = -1$$

$$y = r\sin(\theta) = 2\sin\left(\frac{2\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = z = 1$$

$\Rightarrow$ rectangular coordinates $(x, y, z) = (-1, \sqrt{3}, 1)$.

(b) Find the cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Solution: $r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$

$\tan(\theta) = \frac{y}{x} = \frac{-3}{3} = -1$

$\Rightarrow \quad \theta = 2n\pi - \frac{\pi}{4}$ for integer n.

$z = -7$.

$(n=0)$

So one set of cylindrical coordinates is $(3\sqrt{2}, -\frac{\pi}{4}, -7)$, another is $(3\sqrt{2}, \frac{7\pi}{4}, -7)$. ← $(n=1)$

You may also write $(3\sqrt{2}, \frac{15\pi}{4}, -7)$. ⋯
There are infinitely many choices.

2. Equations of surfaces in polar coordinates. Sometimes maybe strange. Note that if there is not a "z" in the equation, it means "z" can be anything.

Examples: (a) $z = x^2 + y^2$ becomes $z = r^2$

No "θ" in the equation, so "θ" can be anything.

(b) $z = \sqrt{x^2+y^2}$ becomes $z = r$

(c) $x^2 + y^2 + z^2 = 4$ becomes $r^2 + z^2 = 4$.

The top half of the sphere $x^2+y^2+z^2=4$ above xy-plane is $z = \sqrt{4-r^2}$. Of course the bottom half is $z = -\sqrt{4-r^2}$

(d) $z = 3 + x - y$ becomes $z = 3 + r\cos(\theta) - r\sin(\theta)$

(e) $\underbrace{x^2 + y^2 = 1}$ becomes $r^2 = 1$, i.e. $r = 1$.

"z" can be anything. this is a cylinder.

(f) $x^2 - x + y^2 + z^2 = 1$ becomes

$$(r\cos(\theta))^2 - r\cos(\theta) + (r\sin(\theta))^2 + z^2 = 1$$

$$r^2 - r\cos(\theta) + z^2 = 1.$$

(g) $x^2 + (y-1)^2 = 1$ becomes $r = 2\sin(\theta)$

"z" can be anything in the equations. this is a cylinder.

3. If the solid region E is described as

$$E = \{(x, y, z) \mid (x, y) \in D,\ \underbrace{u_1(x, y)}_{\text{floor}} \le z \le \underbrace{u_2(x, y)}_{\text{ceiling}}\}$$

and D in polar is described as

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}.$$

then

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos(\theta),\, r\sin(\theta))}^{u_2(r\cos(\theta),\, r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z)\, r \; dz\, dr\, d\theta$$

Here we rewrite the functions u_1, u_2 in terms of r and θ, and also rewrite function f in terms of r, θ and z.
Additional r in the formula like polar!

Text-Ex 3: A solid E lies between $z = 4$ and $z = 1 - x^2 - y^2$ and within the cylinder $x^2 + y^2 = 1$. The density function is $f(x, y, z) = K\sqrt{x^2+y^2}$. Find the mass of E.

Solution: E in rectangular:

$$\underbrace{x^2 + y^2 \le 1}, \quad 1 - x^2 - y^2 \le z \le 4.$$

Equation for D in xy-plane

Then D can be described as

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 1.$$

and hence E in cylindrical:

$$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ \underline{1 - r^2} \le z \le 4\}.$$

Here we rewrite the equation for the floor in terms of r and θ.

Rewrite $f(x, y, z)$ in terms of r, θ, z:

$$k\sqrt{x^2+y^2} \longrightarrow kr \left(= f(r\cos(\theta), r\sin(\theta), z)\right)$$

So $$m = \iiint_E f(x, y, z)\, dV$$

$$= \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (kr)\, r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 k r^2 \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 kr^2\left[4-(1-r^2)\right] dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 kr^2(3+r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 3kr^2 + kr^4 \, dr\, d\theta$$

$$= \int_0^{2\pi} \left[3k\frac{r^3}{3} + k\frac{r^5}{5}\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} k + \frac{k}{5}\, d\theta$$

$$= \left(k+\frac{k}{5}\right)(2\pi - 0) = \boxed{\frac{12\pi k}{5}}$$

Text- Ex 4 Evaluate

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx$$

by rewriting the integral using cylindrical coordinates.

<u>Solution</u>: Need to find the solid region E first.

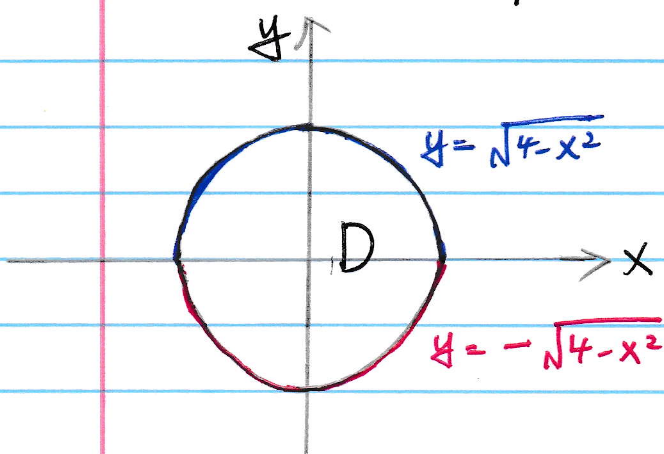
From the iterated integral

$$E = \{ (x, y, z) \mid -2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2},\ \sqrt{x^2+y^2} \le z \le 2 \}$$

(From outside to inside in the iterated integral)

So the region D on xy-plane is

$$D = \{ (x, y) \mid -2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \}$$



D in polar:

$0 \le \theta \le 2\pi$, $0 \le r \le 2$.

Rewrite $\sqrt{x^2+y^2} \le z \le 2$ to get

So E in cylindrical:

$$E = \{ (r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ r \le z \le 2 \}$$

So

The iterated integral in the problem $= \iiint_E (x^2 + y^2)\, dV$

$$= \int_0^{2\pi} \int_0^2 \int_r^2 r^2 \, (r) \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \int_r^2 r^3 \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^3 (2 - r) \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 2r^3 - r^4 \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{2r^4}{4} - \frac{r^5}{5} \right]_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} \left(\frac{2(2)^4}{4} - \frac{2^5}{5} \right) d\theta$$

$$= \int_0^{2\pi} \frac{8}{5} \, d\theta$$

$$= \frac{8}{5} (2\pi - 0) = \boxed{\frac{16\pi}{5}}$$

__Example:__ Find the volume of the solid inside $r = 2\sin(\theta)$, below $z = 9 - x^2 - y^2$ and above the xy-plane ($\leftarrow z = 0$)

__Solution:__ This solid region E has the projection D on xy-plane described as

$$0 \le \theta \le \pi, \quad 0 \le r \le 2\sin(\theta).$$

Rewrite the restriction on z : $0 \le z \le 9 - x^2 - y^2$

as $0 \le z \le 9 - r^2$, then

$$\text{Volume} = \iiint_E 1 \, dV$$

$$= \int_0^{\pi} \int_0^{2\sin(\theta)} \int_0^{9-r^2} 1 \, (r) \, dz \, dr \, d\theta$$

$$= \int_0^{\pi} \int_0^{2\sin(\theta)} (9-r^2) r \, dr \, d\theta$$

$$= \int_0^{\pi} \int_0^{2\sin(\theta)} 9r - r^3 \, dr \, d\theta$$

$$= \int_0^{\pi} \left[\frac{9}{2} r^2 - \frac{1}{4} r^4 \right]_{r=0}^{r=2\sin(\theta)} d\theta$$

$$= \int_0^{\pi} \frac{9}{2} (2\sin(\theta))^2 - \frac{1}{4} (2\sin(\theta))^4 \, d\theta$$

$$= \int_0^{\pi} 18 \sin^2(\theta) - 4 \sin^4(\theta) \, d\theta$$

$$= \int_0^{\pi} 18 \, \frac{1 - \cos(2\theta)}{2} - 4 \left(\frac{1 - \cos(2\theta)}{2} \right)^2 d\theta$$

$$= \int_0^{\pi} 9 - 9\cos(2\theta) - 1 + 2\cos(2\theta) - \frac{1 + \cos(4\theta)}{2} \, d\theta$$

$$= \cdots = \boxed{\frac{15\pi}{2}}$$