15.6 Triple Integrals

1. Introduction: If E is a solid and $f(x, y, z)$ is the density function, then

$$\text{Mass of } E = \iiint_E f(x, y, z)\, dV$$

One special case is $f(x, y, z) = 1$, so

$$\text{Volume of } E = \text{Mass of } E = \iiint_E 1\, dV.$$

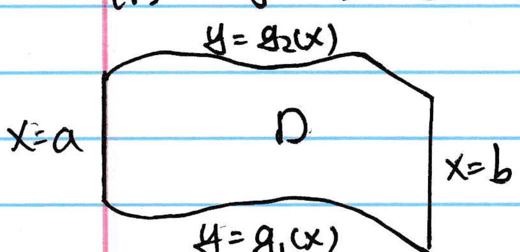
2. How to write triple integrals as iterated integrals for a solid region E of type 1.

E is the region between $u_1(x, y)$ and $u_2(x, y)$
(See Figure 2 in the textbook)

$$E = \{(x, y, z) : (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$$

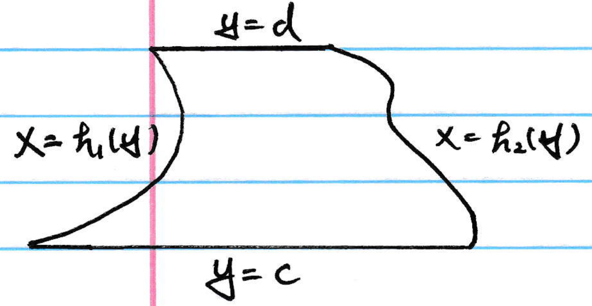
Then
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

(1) If D is vertically simple, then



$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, \underline{dz}\, dy\, dx$$

(2) If D is horizontally simple, then



$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

Text-Ex 1: Evaluate the triple integral $\iiint_B xyz^2\,dV$ where B is the rectangular box

$$B = \{(x,y,z) \mid \underbrace{0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3}\}$$

All the bounds here are constants

Similar to the case of double integrals over a rectangle, we can set up the iterated integral

$$\iiint_B xyz^2\,dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\,dx\,dy\,dz .$$

Note that we can also write

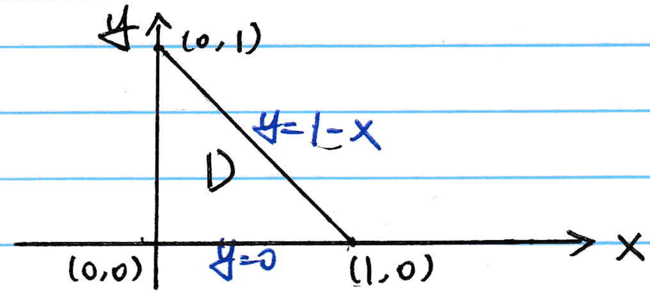
$$\iiint_B xyz^2\,dV = \int_0^1 \int_{-1}^2 \int_0^3 xyz^2\,dz\,dy\,dx ,$$

and this gives us the same answer.

In fact, we have $3 \times 2 = 6$ possible orders of integration and we can use any of them.

Example: Find the integral $\iiint_E xz\, dV$ where E is the solid region between $z = x^2+y^2$ and $z = 1+x^2+y^2$ and above the triangle D in the xy-plane with corners $(0,0)$, $(0,1)$, $(1,0)$.

Solution:



Treat D as a vertically simple region:

$$0 \le x \le 1, \quad 0 \le y \le 1-x.$$

Then

$$\iiint_E xz\, dV$$

$$= \int_0^1 \int_0^{1-x} \int_{x^2+y^2}^{1+x^2+y^2} xz \; dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[x\frac{z^2}{2} \right]_{z=x^2+y^2}^{z=1+x^2+y^2} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[x\frac{(1+x^2+y^2)^2}{2} - x\frac{(x^2+y^2)^2}{2} \right] dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{x}{2} + x(x^2+y^2)\; dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{x}{2} + x^3 + xy^2 \; dy\, dx$$

$$= \int_0^1 \left[\frac{x}{2} y + x^3 y + x \frac{y^3}{3} \right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \frac{x}{2}(1-x) + x^3(1-x) + x \frac{(1-x)^3}{3} \, dx$$

$$= \int_0^1 -\frac{4}{3} x^4 + 2x^3 - \frac{3}{2} x^2 + \frac{5}{6} x \, dx$$

$$= \left[-\frac{4}{3} \frac{x^5}{5} + \frac{x^4}{2} - \frac{1}{2} x^3 + \frac{5}{12} x^2 \right]_0^1$$

$$= \frac{3}{20}.$$

We can also treat D as a horizontally simple region

$$0 \le y \le 1, \quad 0 \le x \le 1-y.$$

Then:

$$\iiint_E xz \, dV$$

$$= \int_0^1 \int_0^{1-y} \int_{x^2+y^2}^{1+x^2+y^2} xz \, dz \, dx \, dy$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

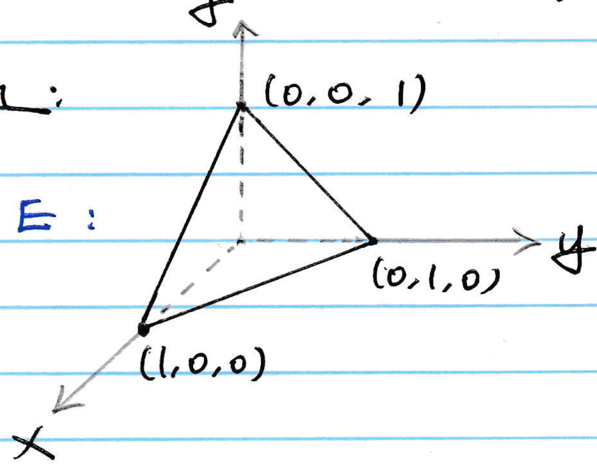
$$= \frac{3}{20}.$$

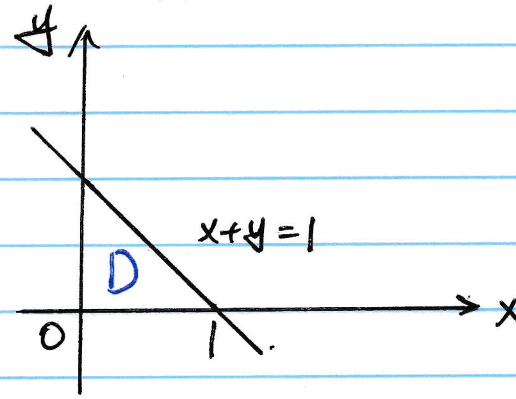
Sometimes, D, $u_1(x, y)$, $u_2(x, y)$ may not be given explicitly in the problem. We need to find them.

Text - Ex 2: Evaluate $\iiint_E z \, dV$ where E is the solid tetrahedron bounded by $x=0$, $y=0$, $z=0$, $x+y+z=1$.

Solution:

E :





Find the intersection between $z=0$ and $x+y+z=1$:

$$x+y=1.$$

So region D on the xy-plane is bounded by

$$x=0, \quad y=0, \quad x+y=1.$$

And we know E is above D and bounded between

$$z=0 \quad \text{and} \quad z=1-x-y \quad (\text{rewrite } x+y+z=1).$$

So by treating D as a vertically simple region:

$$0 \le x \le 1, \qquad 0 \le y \le 1-x,$$

we have

$$\iiint_E z \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx$$

$$= \cdots = \frac{1}{24}.$$

3. Solid region E of type 2, 3.

(1) Type 2 solid region E :

$$E = \{ (x, y, z) \mid (y, z) \in D,\ u_1(y, z) \le x \le u_2(y, z) \}$$

(2) Type 3 Solid region E :

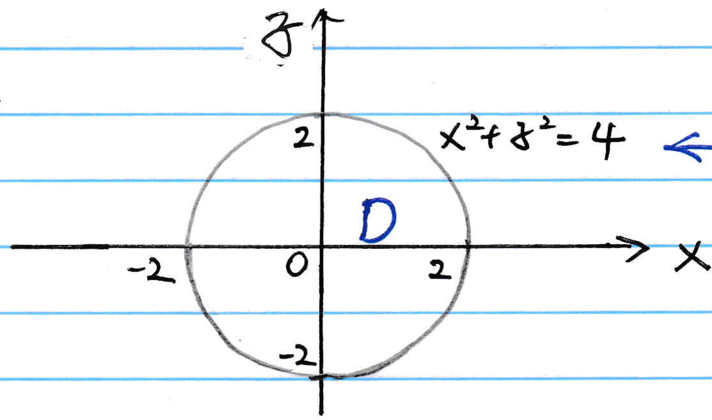
$$E = \{ (x, y, z) \mid (x, z) \in D,\ u_1(x, z) \le y \le u_2(x, z) \}$$

For these two cases, we can also write integration in E as iterated integrals by treating D vertically simple or horizontally simple.

Text - Ex 3 : Evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$

where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Solution :



Find the intersection between $y = x^2+z^2$ and $y = 4$:

$x^2 + z^2 = 4$.

Treat E as as a type 3 solid region b/c we are given two equations $y = u_1(x, z)$, $y = u_2(x, z)$.

If we treat D as vertically simple

$$\iiint_E \sqrt{x^2+z^2}\, dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dz\, dx .$$

If we treat D as horizontally simple

$$\iiint_E \sqrt{x^2+z^2}\, dV = \int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dx\, dz .$$

In fact, the best way is to use polar coordinates
(we will discuss in section 15.7 with details)

$$\iiint_E \sqrt{x^2+z^2}\, dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^{4} r\,(r)\, dy\, dr\, d\theta$$

$$(\; x = r\cos(\theta) \;,\; y = r\sin(\theta) \;)$$

$$= \cdots = \frac{128\pi}{15}$$

4. Applications of Triple Integrals

Total mass in E $\rightarrow$ $m = \iiint_E \underbrace{\rho(x, y, z)}_{\text{density function}} dV$

Center of mass $(\bar{x}, \bar{y}, \bar{z})$:

$$\bar{x} = \frac{1}{m} \iiint_E x\, \rho(x, y, z)\, dV$$

$$\bar{y} = \frac{1}{m} \iiint_E y\, \rho(x, y, z)\, dV$$

$$\bar{z} = \frac{1}{m} \iiint_E z\, \rho(x, y, z)\, dV$$