15.4 Applications of Double Integral

1. Volume of a solid in 3D

When we introduce the double integral, we argue that

$$\text{Volume under graph of } f(x,y) = \iint_R f(x,y)\, dA$$

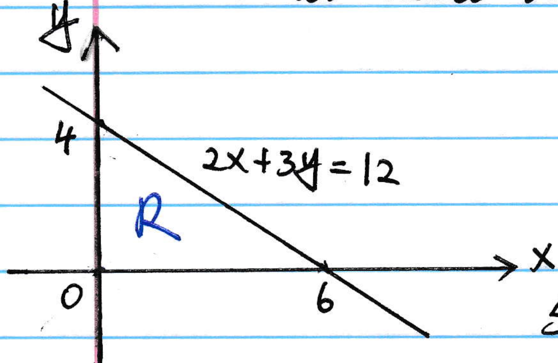
R is usually not explicitly given and we need to find it.

Example (1) Find the volume of solid below the plane $2x+3y+z=12$ and in the first octant.

$\{(x,y,z) : x, y, z \geq 0\}$

Solution: What is R?

We do not need to draw the solid in 3D.



Find the intersection of $2x+3y+z=12$ with the xy-plane by setting $z=0$.

$$2x+3y=12$$

So $R = \{(x,y) : x \geq 0,\ y \geq 0,\ 2x+3y \leq 12\}$

How to set up the integral for the volume?

Rewrite $2x+3y+z=12$ as $z = 12-2x-3y$

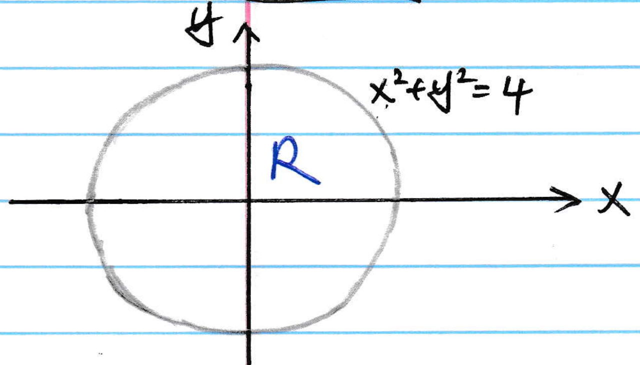
(Note that the points (x,y,z) in the solid satisfy $0 \leq z \leq 12-2x-3y$)

$$\text{Volume} = \iint_R (12-2x-3y)\, dA = \int_0^6 \int_0^{4-\frac{2}{3}x} (12-2x-3y)\, dy\, dx$$

We omit the calculation here.

(2) Find the volume of solid below the paraboloid $z = 4 - x^2 - y^2$ and above the xy-plane $(z=0)$.

Solution: Setting $z=0$ in $z = 4 - x^2 - y^2$ we have $x^2 + y^2 = 4$.



So R is the region

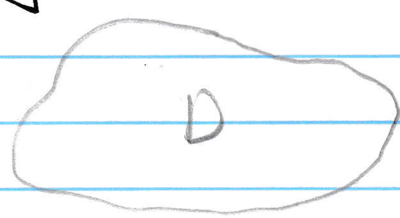
$$R = \{(x, y) : x^2 + y^2 \le 4\}.$$

$$\text{Volume} = \iint_R (4 - x^2 - y^2)\, dA$$

$$\text{(polar coordinates)} = \int_0^{2\pi} \int_0^2 (4 - r^2)\, \underline{r}\, dr\, d\theta .$$

2. Density and Mass

Lamina



Density function $\rho(x, y)$

$$m = \iint_D \rho(x, y)\, dA$$

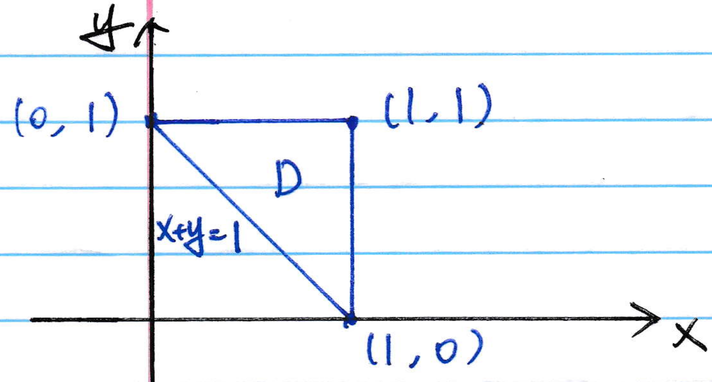
mass of lamina that occupies a region D

Similarly, if $\sigma(x, y)$ is the charge desity, then

$$\text{total charge in } D \longrightarrow Q = \iint_D \sigma(x, y)\, dA$$

Text-Ex 1: Region D is a triangular region with vertices (1, 0), (1, 1) and (0, 1). Charge density is $\sigma(x, y) = xy$, find the total charge.

Solution: Total charge $Q = \iint_D \sigma(x, y)\, dA$



Treat D as a vertically simple region.

$$0 \le x \le 1 \ , \quad 1 - x \le y \le 1$$

$$Q = \int_0^1 \int_{1-x}^1 xy \, dy \, dx$$

$$= \int_0^1 \left[x \frac{y^2}{2} \right]_{y=1-x}^{y=1} dx$$

$$= \int_0^1 \left(x \frac{1^2}{2} - x \frac{(1-x)^2}{2} \right) dx$$

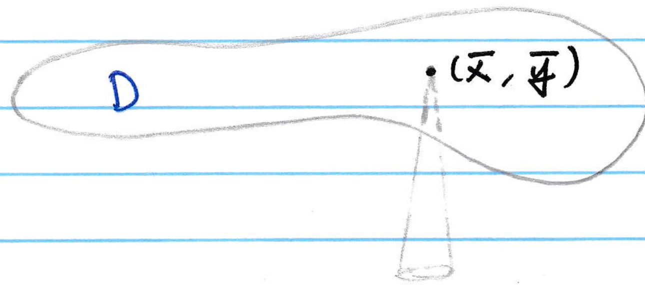
$$= \int_0^1 \frac{1}{2} \left(x - (x - 2x^2 + x^3) \right) dx$$

$$= \frac{1}{2} \int_0^1 2x^2 - x^3 \, dx$$

$$= \frac{1}{2} \left[\frac{2}{3} x^2 - \frac{1}{4} x^4 \right]_0^1$$

$$= \frac{1}{2} \left[\frac{2}{3} (1)^2 - \frac{1}{4} (1)^4 \right] = \frac{1}{2} \left(\frac{5}{12} \right) = \frac{5}{24} .$$

3. Moments and Centers of Mass



lamina D

density function $\rho(x, y)$

Moment about the x-axis : $M_x = \iint_D \rho(x, y)\, y \, dA$

Moment about the y-axis : $M_y = \iint_D \rho(x, y)\, x \, dA$

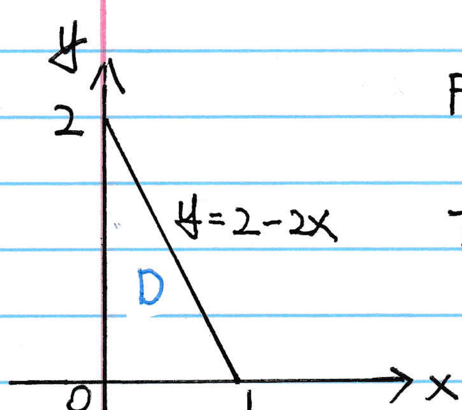
Then the center of mass is

$(\bar{x}, \bar{y})$ where $\bar{x} = \dfrac{M_y}{m} = \dfrac{1}{m} \iint_D \rho(x, y)\, x \, dA$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D \rho(x, y)\, y \, dA$$

(m is the mass $\quad m = \iint_D \rho(x, y) \, dA$)

Text-Ex 2 : Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$ and $(0,2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.



First find mass $m = \iint_D 1 + 3x + y \, dA$

Treat D as a vertically simple region

$$0 \le x \le 1, \quad 0 \le y \le 2 - 2x$$

$$m = \int_0^1 \int_0^{2-2x} (1+3x+y)\, dy\, dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= \int_0^1 \left[(2-2x) + 3x(2-2x) + \frac{(2-2x)^2}{2} - 0 \right] dx$$

$$= \int_0^1 4 - 4x^2\, dx = \left[4x - \frac{4}{3}x^3 \right]_0^1 = \frac{8}{3}.$$

$$M_x = \iint_D \rho(x,y)\, y\, dA = \int_0^1 \int_0^{2-2x} (1+3x+y)\, y\, dy\, dx$$

$$= \int_0^1 \int_0^{2-2x} y + 3xy + y^2\, dy\, dx$$

$$= \int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3} \right]_{y=0}^{y=2-2x} dx$$

$$= \int_0^1 \left[\frac{(2-2x)^2}{2} + 3x\frac{(2-2x)^2}{2} + \frac{(2-2x)^3}{3} \right] dx$$

$$= \int_0^1 \frac{14}{3} - 6x - 2x^2 + \frac{10}{3}x^3\, dx = \frac{11}{6}$$

$$M_y = \iint_D \rho(x,y)\, x\, dA = \int_0^1 \int_0^{2-2x} (1+3x+y)\, x\, dy\, dx$$

$$= \int_0^1 \int_0^{2-2x} x + 3x^2 + xy\, dy\, dx$$

$$= \int_0^1 \left[xy + 3x^2 y + x\frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= \int_0^1 x(2-2x) + 3x^2(2-2x) + x\frac{(2-2x)^2}{2}\, dx$$

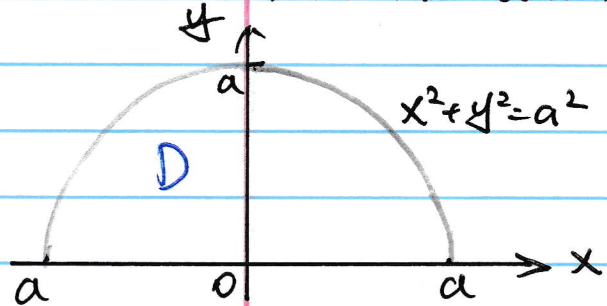
$$= \int_0^1 4x - 4x^3\, dx = \left[2x^2 - x^4 \right]_0^1 = 1.$$

So $\bar{x} = \frac{M_y}{m} = \frac{1}{\frac{8}{3}} = \frac{3}{8}$

$\bar{y} = \frac{M_x}{m} = \frac{\frac{11}{6}}{\frac{8}{3}} = \frac{11}{16}$

$\Rightarrow$ Center of mass $\left(\frac{3}{8}, \frac{11}{16}\right)$.

Text-Ex 3 : Let D be the upper half of the circle $x^2+y^2=a^2$ and density function be $\rho(x,y) = k\sqrt{x^2+y^2}$. Find the center of mass.



$m = \iint_D k\sqrt{x^2+y^2}\, dA$

Use polar coordinates for the integral

$0 \le \theta \le \pi, \quad 0 \le r \le a.$

$$m = \int_0^{\pi}\int_0^{a} kr\,(r)\,dr\,d\theta$$

$$= \int_0^{\pi} \left[k\frac{r^3}{3}\right]_{r=0}^{r=a} d\theta = \int_0^{\pi} k\frac{a^3}{3}\,d\theta$$

$$= \left[k\frac{a^3}{3}\theta\right]_0^{\pi} = \frac{k\pi a^3}{3}.$$

So $\bar{x} = \frac{1}{m}\underbrace{\iint_D k\sqrt{x^2+y^2}\,(x)\,dA}_{=\,0}$ by symmetry w.r.t y-axis

$\bar{y} = \frac{1}{m}\iint_D k\sqrt{x^2+y^2}\,(y)\,dA$

$= \frac{1}{m}\int_0^{\pi}\int_0^{a} kr\,(r\sin(\theta))\,r\,dr\,d\theta$

$$= \frac{1}{m} \int_0^{\pi} \int_0^{a} k\, r^3 \sin(\theta)\, dr\, d\theta$$

$$= \frac{1}{m} \int_0^{\pi} \left[k \frac{r^4}{4} \sin(\theta) \right]_{r=0}^{r=a} d\theta$$

$$= \frac{1}{m} \int_0^{\pi} k \frac{a^4}{4} \sin(\theta)\, d\theta$$

$$= \frac{ka^4}{4m} \int_0^{\pi} \sin(\theta)\, d\theta$$

$$= \frac{ka^4}{4m} \left[-\cos(\theta) \right]_0^{\pi} = \frac{ka^4}{4m} \left(1 - (-1)\right)$$

$$= \frac{ka^4}{4 \frac{k\pi a^3}{3}} (2) = \frac{3a}{2\pi}.$$

So the center of mass is $(\bar{x}, \bar{y}) = \left(0, \frac{3a}{2\pi}\right)$.

4. Moment of Inertia

Moment of inertia about the x-axis

$$I_x = \iint_D y^2 \rho(x, y)\, dA$$

Moment of inertia about the y-axis

$$I_y = \iint_D x^2 \rho(x, y)\, dA$$

Moment of inertia about the origin (polar moment of inertia)

$$I_0 = \iint_D (x^2 + y^2) \rho(x, y)\, dA = I_x + I_y$$

Text-Ex 4: Find I_x, I_y and I_0 for a disk D with radius a, constant density $\rho(x, y) = \rho$.

Solution: Use polar coordinates.

$$I_x = \iint_D y^2 \rho(x, y)\, dA = \int_0^{2\pi}\int_0^a (r\sin(\theta))^2 \rho\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^a r^3 \sin^2(\theta)\,\rho\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4}\sin^2(\theta)\,\rho\right]_{r=0}^{r=a} d\theta$$

$$= \int_0^{2\pi} \frac{a^4}{4}\sin^2(\theta)\,\rho\, d\theta$$

$$= \frac{a^4\rho}{4}\int_0^{2\pi}\sin^2(\theta)\,d\theta = \frac{a^4\rho}{4}\int_0^{2\pi}\frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= \frac{a^4\rho}{4}\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{2\pi} = \frac{a^4\rho\pi}{4}.$$

Similarly (or by symmetry $I_y = I_x$), $I_y = \frac{a^4\rho\pi}{4}$.

So $I_0 = I_x + I_y = \frac{a^4\rho\pi}{2}$.

Definition of radius of gyration

Radius of gyration about y-axis: $\bar{\bar{x}}$ satisfying $m\bar{\bar{x}}^2 = I_y$

Radius of gyration about x-axis: $\bar{\bar{y}}$ satisfying $m\bar{\bar{y}}^2 = I_x$

One can also write

$$\bar{\bar{x}}^2 = \frac{1}{m} \iint_D x^2 \rho(x, y)\, dA$$

$$\bar{\bar{y}}^2 = \frac{1}{m} \iint_D y^2 \rho(x, y)\, dA$$

<u>Text-Ex 5</u>: Find the radius of gyration about the x-axis of the disk in Text-Ex 4.

<u>Solution</u>:

$$\bar{\bar{y}}^2 = \frac{1}{m} \iint_D y^2 \rho(x, y)\, dA$$

In Text-Ex 4, we have

$$I_x = \iint_D y^2 \rho(x, y)\, dA = \frac{a^4 \rho \pi}{4}.$$

And the mass of the disk is

$$m = \text{Area(disk)}\, \rho = \pi a^2 \rho.$$

So

$$\bar{\bar{y}}^2 = \frac{(a^4 \rho \pi)/4}{\pi a^2 \rho} = \frac{a^2}{4}$$

$$\Rightarrow \quad \bar{\bar{y}} = \frac{a}{2}.$$