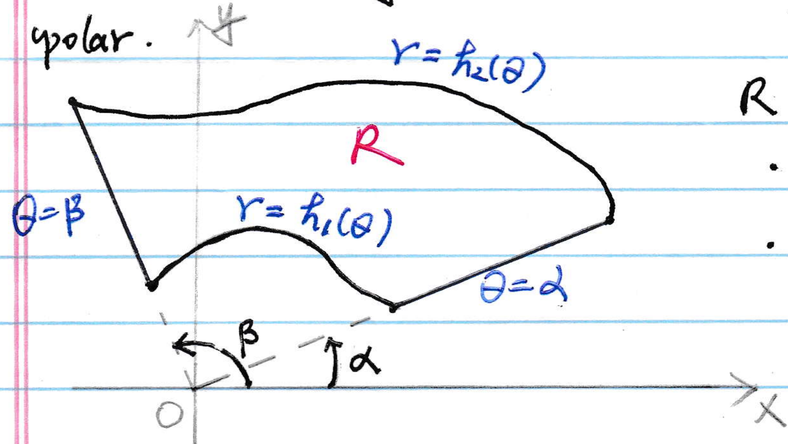
15.3 Double Integrals in Polar Coordinates

1. Recall that the polar coordinates (r, θ) of a point (x, y) satisfy

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

Sometimes a region R can be easier to describe using polar.

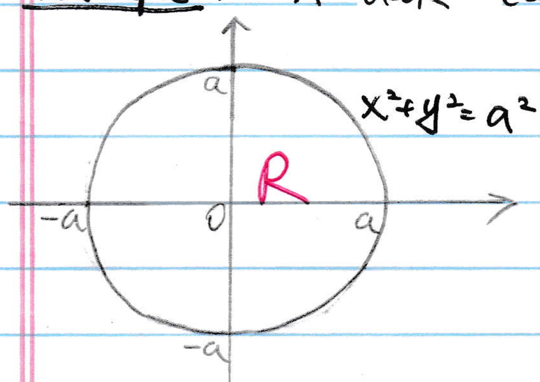


R in the plot is between

- two angles $\theta = \alpha$ and $\theta = \beta$
- two functions $r = h_1(\theta)$ and $r = h_2(\theta)$.

2. Examples of some regions R in polar coordinates.

Example: A disk centered at origin with radius a.



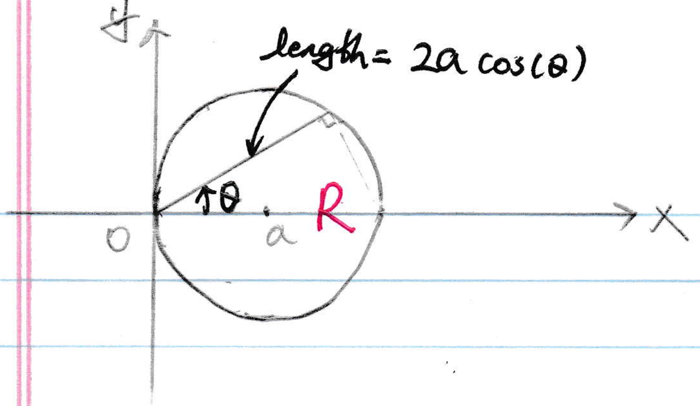
Notice that $x^2 + y^2 = r^2$.

So in polar coordinates, R is:

$$0 \le r \le a, \quad 0 \le \theta \le 2\pi.$$

Example: A disk centered at $(a, 0)$ with radius a, i.e. region inside a circle with equation $(a > 0)$

$$(x - a)^2 + y^2 = a^2.$$



Notice that the equation can be rewritten as :

$$x^2 - 2ax + a^2 + y^2 = a^2$$

$$\underbrace{x^2 + y^2}_{= r^2} - 2a\underbrace{x}_{= r\cos(\theta)} = 0$$

$$r^2 - 2ar\cos(\theta) = 0$$

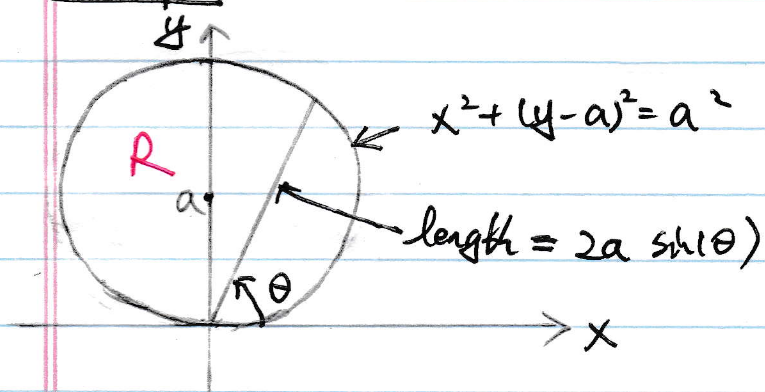
$$\Rightarrow \quad r = 2a\cos(\theta) \quad (\text{or } r = 0).$$

↳ Only refers to the origin.

So R can be represented by

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2a\cos(\theta)$$

<u>Example</u> : A disk centered at $(0, a)$ with radius $a > 0$.



$$x^2 + (y-a)^2 = a^2$$

$$x^2 + y^2 - 2ay + a^2 = a^2$$

$$r^2 - 2ar\sin(\theta) = 0$$

$$\Rightarrow r = 2a\sin(\theta) \quad (\text{or } r = 0)$$

So R can be represented by

$$0 \le \theta \le \pi, \quad 0 \le r \le 2a\sin(\theta).$$

3. If R is described in polar coordinates by

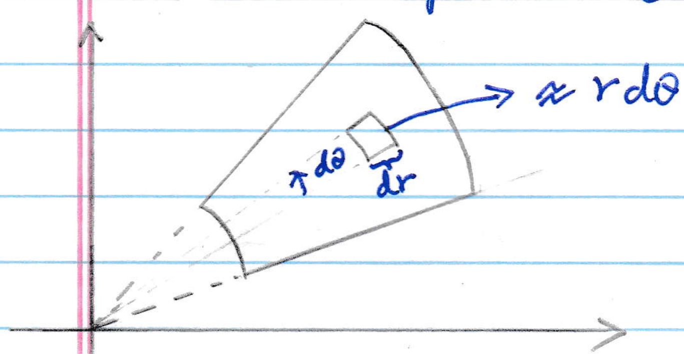
$$\alpha \le \theta \le \beta \;, \qquad h_1(\theta) \le r \le h_2(\theta).$$

Then:
$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \underline{f(r\cos(\theta), r\sin(\theta))}\, r \, dr\, d\theta$$

"$f(r\cos(\theta), r\sin(\theta))$" means converting f into polar coordinates

Remark: Do not forget the additional r. Section 15.9 has a rigorous explanation of this.

An intuitive explanation (not required)

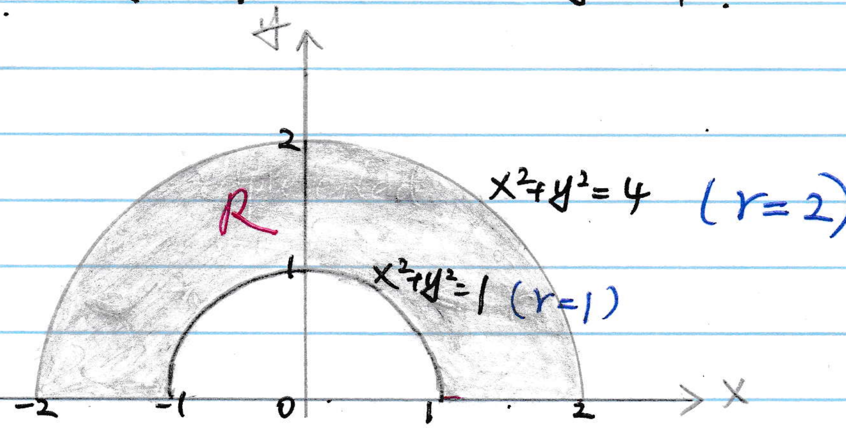


$\Rightarrow$ The area of that small piece is $\approx (r\, d\theta)(dr) = r\, d\theta\, dr$.

(See the textbook for details)

Text-Ex 1: Evaluate $\iint_R (3x + 4y^2)\, dA$ where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution:



We know $x^2 + y^2 = 4$ in polar is $r = 2$,

$x^2 + y^2 = 1$ in polar is $r = 1$,

and the upper half-plane ($y \geq 0$) in polar is $0 \leq \theta \leq \pi$

because

$$r \sin(\theta) = y \geq 0$$

$$\Rightarrow \quad \theta \in [0, \pi].$$

So in polar R is given by:

$$0 \leq \theta \leq \pi, \quad 1 \leq r \leq 2.$$

All the bounds for θ, r are constants. This is a special case where R is a polar rectangle. (Not important)

Write $3x + 4y^2$ using polar coordinates (r, θ).

Since $x = r\cos(\theta)$, $y = r\sin(\theta)$,

$$3x + 4y^2 = 3r\cos(\theta) + 4r^2 \sin^2(\theta).$$

Therefore

$$\iint_R (3x + 4y^2)\, dA = \int_0^\pi \int_1^2 \left(3r\cos(\theta) + 4r^2\sin^2(\theta)\right) r\, dr\, d\theta$$

$$= \int_0^\pi \int_1^2 3r^2 \cos(\theta) + 4r^3 \sin^2(\theta))\, dr\, d\theta$$

$$= \int_0^\pi \left[r^3 \cos(\theta) + r^4 \sin^2(\theta)\right]_{r=1}^{r=2} d\theta$$

$$= \int_0^\pi \left[\left(2^3 \cos(\theta) + 2^4 \sin^2(\theta)\right) - \left(\cos(\theta) + \sin(\theta)\right)\right] d\theta$$

$$= \int_0^\pi 7\cos(\theta) + 15 \sin^2(\theta)\, d\theta$$

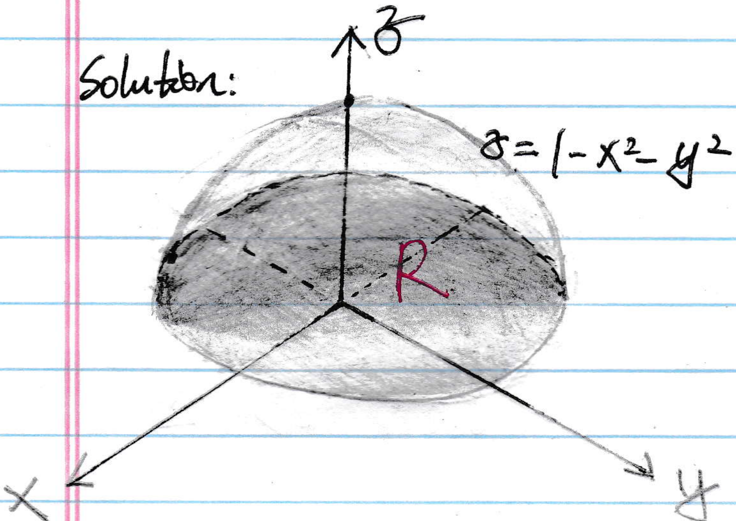
$$= \int_0^{\pi} 7\cos(\theta) + 15\left(\frac{1-\cos(2\theta)}{2}\right) d\theta$$

$$= \left[7\sin(\theta) + 15\left(\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right)\right]_0^{\pi}$$

$$= \boxed{\frac{15\pi}{2}}$$

Text-Ex 2: Find the volume of the solid bounded by the plane $z = 0$, and the paraboloid $z = 1 - x^2 - y^2$.

Solution:



$$0 \le z \le 1 - x^2 - y^2$$

$$\Rightarrow \quad x^2 + y^2 \le 1$$

$$\Rightarrow \quad r^2 \le 1$$

$$0 \le r \le 1$$

So the base of the solid, i.e. region R is

$0 \le \theta \le 2\pi$, $0 \le r \le 1$.

(b/c no restriction on θ)

Therefore

$$\text{Volume} = \iint_R (1 - x^2 - y^2)\, dA = \int_0^{2\pi}\int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r - r^3 \, dr\, d\theta$$

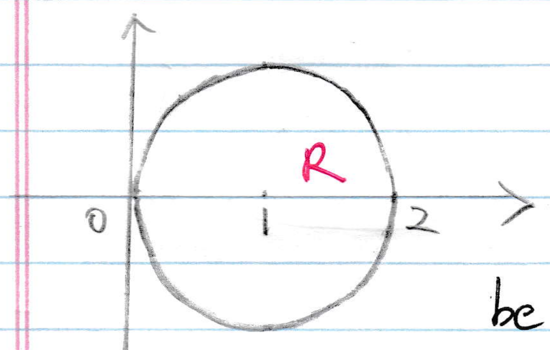
$$= \int_0^{2\pi} \left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4}\right) d\theta = \int_0^{2\pi} \frac{1}{4}\, d\theta$$

$$= \frac{\theta}{4}\Big|_0^{2\pi} = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$$

<u>Text-Ex 4:</u> Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

<u>Solution:</u> Inside $x^2 + y^2 = 2x$ means R is the region in the plot b/c



$$x^2 + y^2 = 2x \iff (x-1)^2 + y^2 = 1.$$

We have seen such a region can be represented by:

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2\cos(\theta).$$

So:

$$\text{Volume} = \iint_R (x^2 + y^2)\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} r^2\, r\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} r^3\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_{r=0}^{r=2\cos(\theta)} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}\left(2\cos(\theta)\right)^4 d\theta$$

I will give you the anti-derivative for this if needed.

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \cos^4(\theta)\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + 2\cos(2\theta) + \cos^2(2\theta)\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + 2\cos(2\theta) + \frac{1+\cos(4\theta)}{2}\, d\theta$$

$$= \left[\theta + \sin(2\theta) + \frac{1}{2}\left(\theta + \frac{1}{4}\sin(4\theta)\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \boxed{\frac{3\pi}{2}}$$

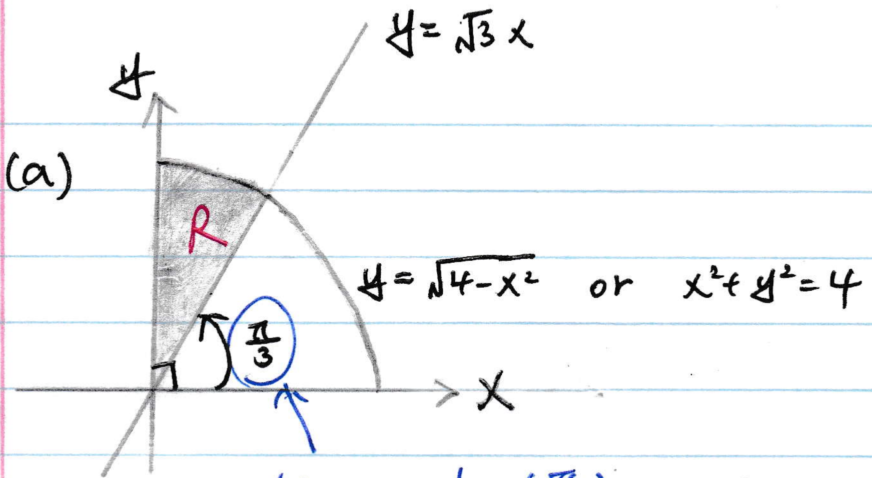
4. About sections 15.2, 15.3: Sometimes an iterated integral has been set up one way (Vertically simple, horizontally simple, polar) and is easier another way. In this case we might rewrite it.

Example: Consider $\displaystyle\int_0^1 \int_{x\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

(a) Draw the picture and find out the region R where we are computing the integral.

(b) Set up the iterated integral in polar coordinates and compute the integral.

(a)



b/c $\tan\left(\frac{\pi}{3}\right) = \sqrt{3} \leftarrow$ slope of $y = \sqrt{3}\,x$.

(b) In polar, R is described by

$$\frac{\pi}{3} \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2.$$

So

$$\int_0^1 \int_{x\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^2 r \; r \, dr\, d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^2 r^2 \, dr\, d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\frac{1}{3} r^3\right]_{r=0}^{r=2} d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{8}{3}\, d\theta = \frac{8}{3}\theta \Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \left(\frac{8}{3}\,\frac{\pi}{2}\right) - \left(\frac{8}{3}\,\frac{\pi}{3}\right) = \boxed{\frac{4\pi}{9}}$$