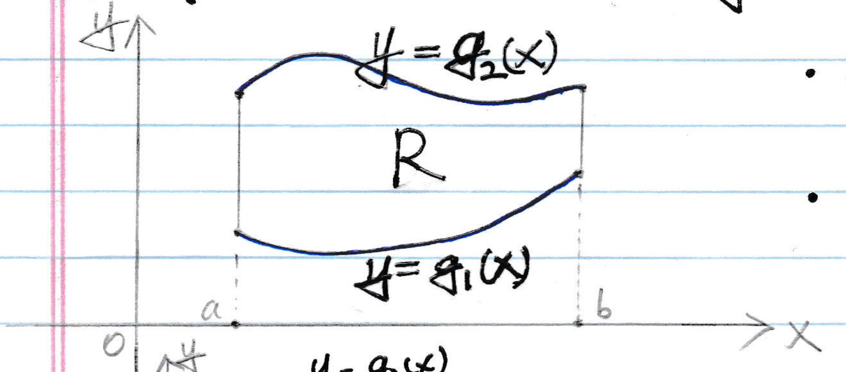
15.2 Double Integrals over General Regions

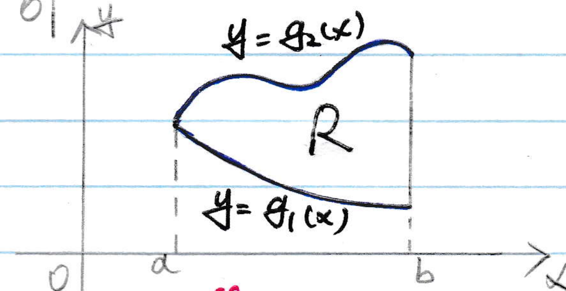
1. Goal: Rewrite double integral $\iint_R f(x,y)\,dA$ as an iterated double integral.

2. Definition: R is vertically simple (type I) if R is

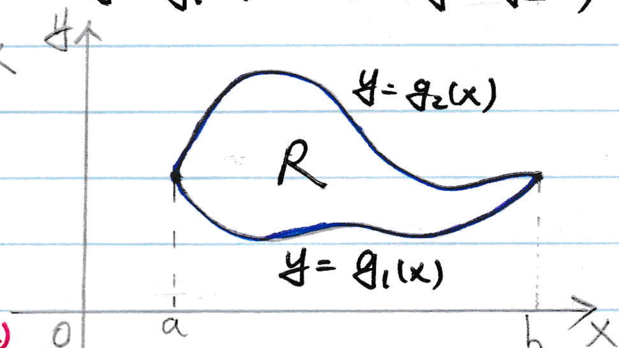


- Between two constant x-values $x = a$ and $x = b$
- Between functions $y = g_1(x)$ and $y = g_2(x)$

or



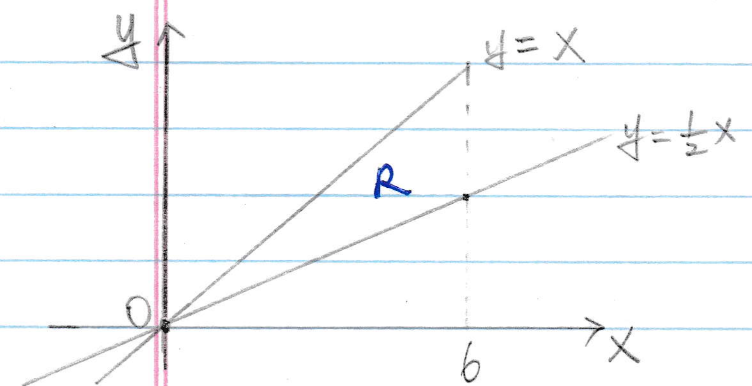
or



Then: $\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$.

Example: Compute $\iint_R xy\,dA$ where R is the region between $y = \frac{1}{2}x$ and $y = x$ and to the left of $x = 6$.

Solution:



$$\iint_R xy\,dA$$

$$= \int_0^6 \int_{\frac{1}{2}x}^{x} xy\,dy\,dx$$

$$\int_0^6 \int_{\frac{1}{2}x}^{x} xy \; dy \; dx$$

$$= \int_0^6 \left[x \tfrac{1}{2} y^2 \right]_{y=\frac{1}{2}x}^{y=x} dx$$

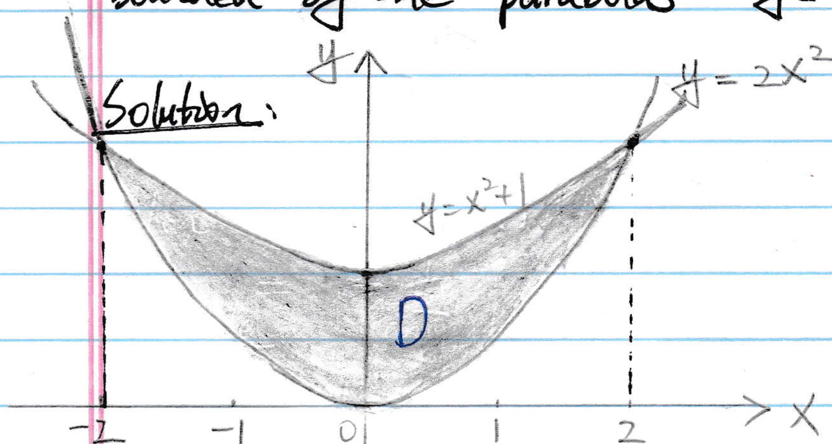
$$= \int_0^6 \left[\left(x \tfrac{1}{2} x^2 \right) - \left(x \tfrac{1}{2} \left(\tfrac{1}{2}x\right)^2 \right) \right] dx$$

$$= \int_0^6 \frac{3}{8} x^3 \, dx = \frac{3}{32} x^4 \Big|_0^6$$

$$= \frac{3}{32}(6)^4 - \frac{3}{32}(0)^4 = \boxed{\frac{243}{2}}$$

Text-Ex 1: Evaluate $\iint_D (x+2y)\,dA$ where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution:



$$\iint_D (x+2y)\,dA = ?$$

Solve intersection points: $x^2 + 1 = 2x^2 \Rightarrow x^2 = 1$
$\Rightarrow x = \pm 1$.

So
$$\iint_D (x+2y)\,dA = \int_{-1}^{1} \int_{2x^2}^{x^2+1} (x+2y)\,dy\,dx$$

$$= \int_{-1}^{1} \left[xy + y^2 \right]_{y=2x^2}^{y=x^2+1} dx$$

$$= \int_{-1}^{1} \left[\left(x(x^2+1) + (x^2+1)^2\right] - \left[x(2x^2) + (2x^2)^2\right] dx$$

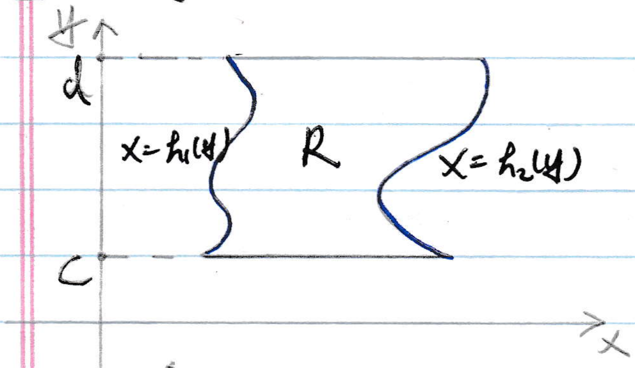
$$= \int_{-1}^{1} \left[x^3 + x + x^4 + 2x^2 + 1 - 2x^3 - 4x^4\right] dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

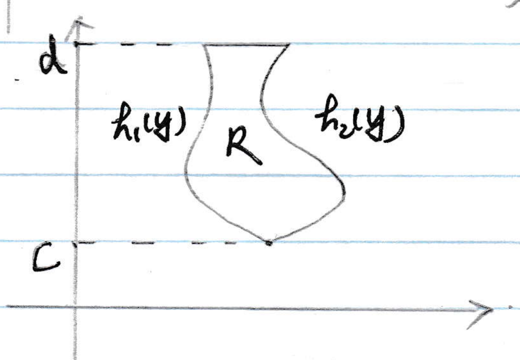
$$= \left(-\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x\right)\Bigg|_{-1}^{1}$$

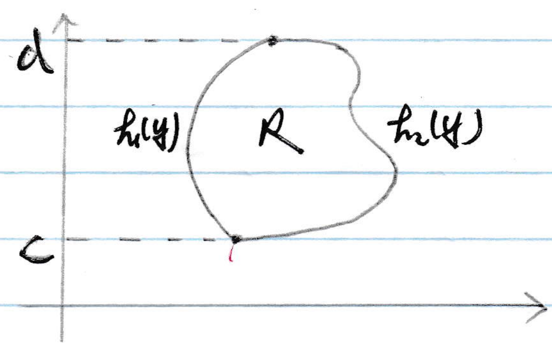
$$= \left(-\frac{3}{5}(1)^5 - \frac{1}{4}(1)^4 + \frac{2}{3}(1)^3 + \frac{1}{2}(1)^2 + 1\right)$$
$$- \left(-\frac{3}{5}(-1)^5 - \frac{1}{4}(-1)^4 + \frac{2}{3}(-1)^3 + \frac{1}{2}(-1)^2 + (-1)\right) = \boxed{\frac{32}{15}}$$

3. Definition: R is horizontally simple if R is (type II)



- Between two constant y-values $y = c$ and $y = d$
- Between functions $x = h_1(y)$ and $x = h_2(y)$.

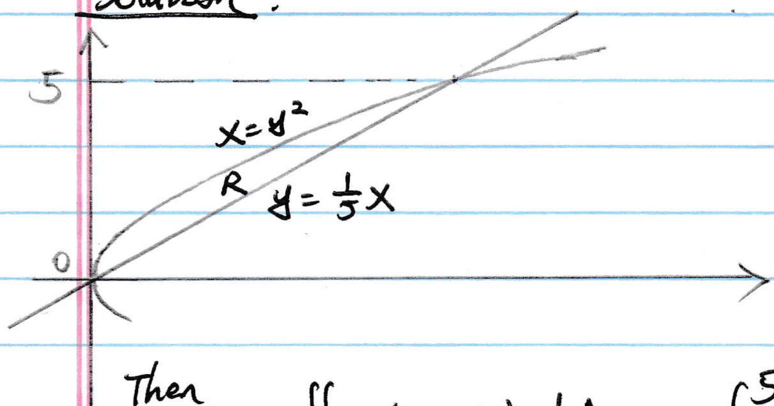




Then: $\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$

Example: Compute $\iint_R (x+y)\,dA$ where R is the region between $x = y^2$ and $y = \frac{1}{5}x$.

Solution:



Find intersection points by solving $\begin{cases} x = y^2 \\ y = \frac{1}{5}x \end{cases}$

$\Rightarrow 5y = y^2$

$\Rightarrow y(y-5) = 0$

$\Rightarrow y = 0$ or 5

Then

$$\iint_R (x+y)\,dA = \int_0^5 \int_{y^2}^{5y} (x+y)\,dx\,dy$$

We rewrite $y = \frac{1}{5}x$ as $x = 5y$.

$$= \int_0^5 \left[\frac{1}{2}x^2 + xy\right]_{x=y^2}^{x=5y} dy$$

$$= \int_0^5 \left[\left(\frac{1}{2}(5y)^2 + (5y)y\right) - \left(\frac{1}{2}(y^2)^2 + (y^2)y\right)\right] dy$$

$$= \int_0^5 \left(\frac{25}{2}y^2 + 5y^2 - \frac{1}{2}y^4 - y^3\right) dy$$

$$= \int_0^5 \frac{35}{2}y^2 - \frac{1}{2}y^4 - y^3\,dy$$

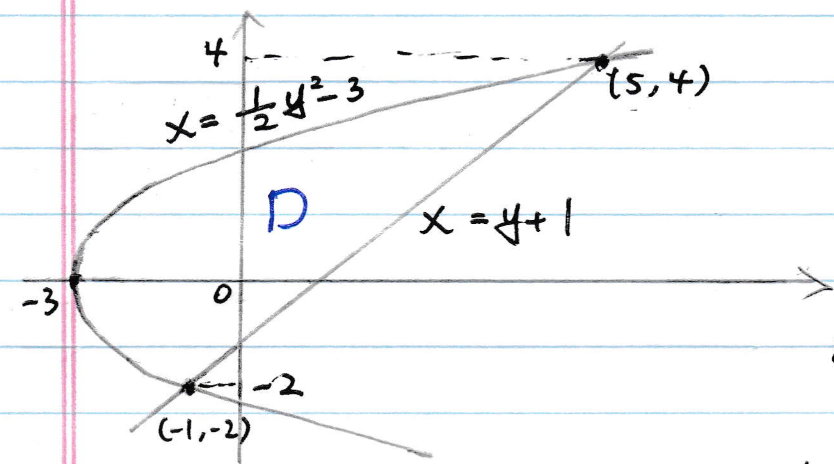
$$= \left(\frac{35}{6}y^3 - \frac{1}{10}y^5 - \frac{1}{4}y^4\right)\Big|_0^5 = \boxed{\frac{3125}{12}}$$

Text-Ex 3: Evaluate $\iint_D xy\,dA$ where D is the region bounded by the line $y = x-1$ and the parabola $y^2 = 2x+6$.

Solution: Find intersection points by solving

$$\begin{cases} y = x - 1 \\ y^2 = 2x + 6 \end{cases} \Rightarrow \text{Substitute } x = y+1: \quad y^2 = 2(y+1) + 6$$

$$\Rightarrow y^2 - 2y - 8 = 0 \Rightarrow (y-4)(y+2) = 0 \Rightarrow y = 4 \text{ or } -2$$



Rewrite

$y = x - 1$ as $x = y + 1$

$y^2 = 2x + 6$ as $x = \frac{1}{2}y^2 - 3$

b/c D is horizontally simple.

Then

$$\iint_D xy \, dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2 - 3}^{y+1} xy \, dx \, dy$$

$$= \int_{-2}^{4} \left[\frac{x^2}{2} y \right]_{x = \frac{1}{2}y^2 - 3}^{x = y+1} dy$$

$$= \int_{-2}^{4} \left[\frac{(y+1)^2}{2} y - \frac{1}{2} \left(\frac{1}{2} y^2 - 3 \right)^2 y \right] dy$$

$$= \int_{-2}^{4} \left(- \frac{y^5}{8} + 2y^3 + y^2 - 4y \right) dy$$

$$= \left[- \frac{1}{48} y^6 + \frac{1}{2} y^4 + \frac{1}{3} y^3 - 2y^2 \right]_{-2}^{4}$$

$$= \boxed{36}$$

4. R is simple if it's both, and can then be done in either way.