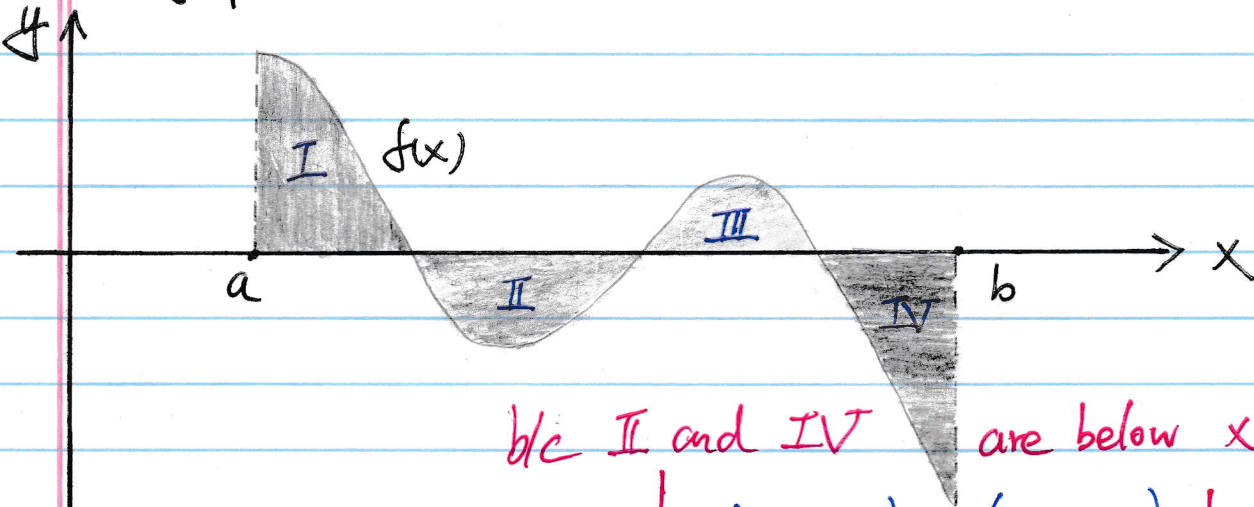
15.1 Double Integrals over Rectangles

1. Recall that $\int_a^b f(x)\,dx$ is (signed) area under the graph of $f(x)$.



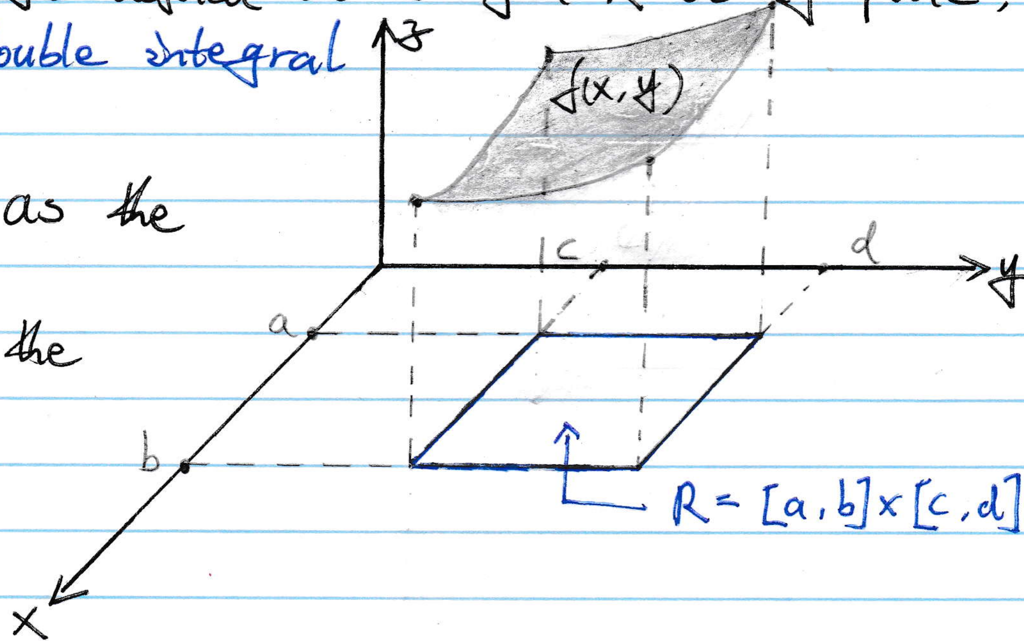
b/c II and IV are below x-axis.

$$\int_a^b f(x)\,dx = \begin{pmatrix}\text{Area of}\\ \text{I}\end{pmatrix} - \begin{pmatrix}\text{Area of}\\ \text{II}\end{pmatrix} + \begin{pmatrix}\text{Area of}\\ \text{III}\end{pmatrix} - \begin{pmatrix}\text{Area of}\\ \text{IV}\end{pmatrix}$$

2. For $f(x,y)$ defined on a region R in xy-plane, define the double integral

$\iint_R f(x,y)\,dA$ as the

(signed) volume under the graph of $f(x,y)$.



This works in general for R not a rectangle.

3. Iterated Integrals: Focus on the notation and process now. We'll discuss the connection between the iterated integrals and the double integrals later.

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \Big[\underbrace{\int_c^d f(x,y)\,dy}_{} \Big]\,dx$$

Compute the integral inside first.

Text-Ex 4 Evaluate the iterated integrals.

(a) $\displaystyle\int_0^3 \int_1^2 x^2 y\,dy\,dx$

Ans: Consider $\displaystyle\int_1^2 x^2 y\,dy$. Treat x as a constant

and obtain

$$\int_1^2 x^2 y\,dy = \left[x^2 \frac{y^2}{2}\right]_{y=1}^{y=2} = \left(x^2 \frac{2^2}{2}\right) - \left(x^2 \frac{1^2}{2}\right)$$

$$= 2x^2 - \frac{1}{2}x^2 = \frac{3}{2}x^2.$$

So the iterated integral

$$\int_0^3 \int_1^2 x^2 y\,dy\,dx = \int_0^3 \frac{3}{2}x^2\,dx = \frac{1}{2}x^3 \Big|_0^3$$

$$= \left(\frac{1}{2}3^3\right) - \left(\frac{1}{2}0^3\right) = \boxed{\frac{27}{2}}$$

(b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Ans: First integrate $\int_0^3 x^2 y \, dx$. (Notice that we'll treat y as a constant when we integrate w.r.t x)

$$\int_0^3 x^2 y \, dx = \left[\frac{x^3}{3} y\right]_{x=0}^{x=3} = \left(\frac{3^3}{3} y\right) - \left(\frac{0^3}{3} y\right)$$

$$= 9y - 0 = 9y.$$

So

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 9y \, dy = \frac{9}{2} y^2 \Big|_1^2$$

$$= \left(\frac{9}{2} 2^2\right) - \left(\frac{9}{2} 1^2\right) = \boxed{\frac{27}{2}}$$

Text-Exercise 15: Compute $\int_1^4 \int_0^2 (6x^2 y - 2x) \, dy \, dx$

Ans: $\int_1^4 \int_0^2 (6x^2 y - 2x) \, dy \, dx$

$$= \int_1^4 \left[(6x^2)\left(\frac{y^2}{2}\right) - 2x y\right]_{y=0}^{y=2} dx$$

$$= \int_1^4 \left[\left(6x^2 \frac{2^2}{2} - 2x(2)\right) - \left(6x^2 \frac{0^2}{2} - 2x(0)\right)\right] dx$$

$$= \int_1^4 12x^2 - 4x \, dx$$

$$= 4x^3 - 2x^2 \Big|_{x=1}^4 = \left(4(4)^3 - 2(4)^2\right) - \left(4(1)^3 - 2(1)^2\right)$$

$$= (256 - 32) - (4 - 2) = \boxed{222}$$

4. Rewrite $\iint_R f(x,y)\,dA$ as an iterated double integral. This process depends on the shape of R. For rectangular R in this section, this is easy.

Fubini's Thm: For $R = [a,b] \times [c,d] = \{(x,y) : a \le x \le b,\ c \le y \le d\}$,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

One can use either of the iterated integrals for calculation. Sometimes one would be easier compared with the other.

Text - Ex 5: For $R = [0,2] \times [1,2]$, compute

$$\iint_R (x - 3y^2)\,dA$$

Solution 1: Use $\iint_R (x-3y^2)\,dA = \int_0^2 \int_1^2 (x-3y^2)\,dy\,dx$.

$$\int_0^2 \int_1^2 (x-3y^2)\,dy\,dx = \int_0^2 \Big[xy - y^3\Big]_{y=1}^{y=2} dx$$

$$= \int_0^2 \Big[(2x-8) - (x-1)\Big]\,dx = \int_0^2 (x-7)\,dx$$

$$= \Big[\frac{x^2}{2} - 7x\Big]_0^2 = (2-14) - (0-0) = \boxed{-12}$$

Solution 2:

$$\iint_R (x-3y^2)\,dA = \int_1^2 \int_0^2 (x - 3y^2)\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2 \right]_{x=0}^{x=2} dy$$

$$= \int_1^2 \left[\left(\frac{2^2}{2} - 3(2)y^2 \right) - \left(\frac{0^2}{2} - 3(0)y^2 \right) \right] dy$$

$$= \int_1^2 (2 - 6y^2)\, dy = \left[2y - 2y^3 \right]_1^2 = \boxed{-12}$$

Text - Ex 6 : Evaluate $\iint_R y \sin(xy)\, dA$ for $R = [1,2] \times [0, \pi]$

Solution : $\iint_R y \sin(xy)\, dA = \int_0^\pi \int_1^2 y \sin(xy)\, dx\, dy$

(Notice that $\int_1^2 y \sin(xy)\, dx = \int_y^{2y} \sin(t)\, dt = \left[-\cos(t) \right]_{t=y}^{t=2y}$
for $t = xy$ substitution)

$$= \int_0^\pi \left[-\cos(xy) \right]_{x=1}^{x=2} dy = \int_0^\pi \left(-\cos(2y) + \cos(y) \right) dy$$

$$= \left[-\frac{1}{2} \sin(2y) + \sin(y) \right]_0^\pi = \boxed{0} .$$

Remark : If we try $\iint_R y \sin(xy)\, dA = \int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx$,
then the integration $\int_0^\pi y \sin(xy)\, dy$ is much more
difficult than $\int_1^2 y \sin(xy)\, dx$ in the solution above.

It is sometimes important to choose the order of integration making calculation easier.