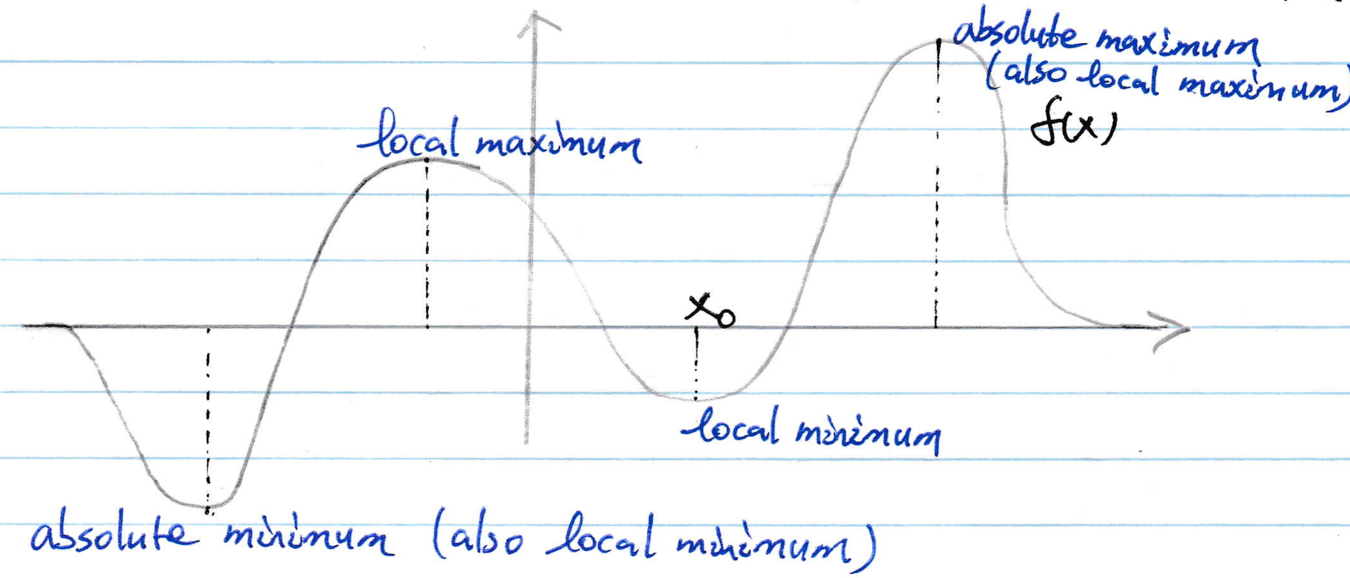
14.7 Maximum and Minimum Values

1. Definitions : absolute maximum , absolute minimum , local maximum , local minimum .



We say $f(x)$ has a local minimum at $x = x_0$ and $f(x_0)$ is a local minimum value .

See Figure 1 in the textbook for formal definitions of absolute/local maximum/minimum for $f(x, y)$.

- $\boxed{\text{Definition of critical points}}$:

Point (x, y) s.t. $\nabla f(x, y) = \vec{0}$

(or at least one partial derivative is undefined)

← We won't use this

Example : Find all critical points of

$$f(x, y) = x^2 + y^2 - 2x - 6y + 14$$

Ans: $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

$\nabla f(x,y) = \langle 2x-2, 2y-6 \rangle$

Solve $\nabla f(x,y) = \langle 2x-2, 2y-6 \rangle = \vec{0}$:

$$\begin{cases} 2x-2=0 \\ 2y-6=0 \end{cases} \implies \begin{cases} x=1 \\ y=3 \end{cases}.$$

So $\boxed{(1,3)}$ is a critical point of f.

2. Finding local max and min.

- Procedure: (1) Find all critical points
  (2) Find the discriminant

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

(here we use $f_{xy} = f_{yx}$, which is almost always true)

(3) For each critical point:

- (a) If $D(x_0, y_0) > 0$ then check $f_{xx}(x_0, y_0)$.
  If $f_{xx}(x_0, y_0)$ positive, then local min
  If $f_{xx}(x_0, y_0)$ negative, then local max
- (b) If $D(x_0, y_0) < 0$ then saddle point

Three outcomes for critical points after Second Derivatives Test: local min, local max, saddle point.

Example: Find the local maximum and minimum values and saddle points of $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

(1) Find critical points

Ans: $\nabla f(x,y) = \langle 6xy - 6x,\ 3x^2 + 3y^2 - 6y \rangle$.

Solve $\nabla f(x,y) = \vec{0}$:

$$\begin{cases} 6xy - 6x = 0 & ① \\ 3x^2 + 3y^2 - 6y = 0 & ② \end{cases}$$

From ①, $6x(y-1) = 0$.

So either $\underline{x = 0}$ or $y = 1$.

← Don't forget this.

If $x = 0$, plug $x = 0$ into ②: $3y^2 - 6y = 0$

$\Rightarrow 3y(y-2) = 0 \Rightarrow y = 0$ or $y = 2$.

So we get two critical points in this case ($x=0$):

$(0,0)$ and $(0,2)$.

If $y = 1$, plug $y = 1$ into ②: $3x^2 - 3 = 0$

$\Rightarrow x^2 = 1 \Rightarrow x = 1$ or $x = -1$.

So two more critical points in this case ($y=1$):

$(1,1)$ and $(-1,1)$.

In total, we have four critical points

$(0,0)$, $(0,2)$, $(1,1)$ and $(-1,1)$.

(2) Find the discriminant $D(x, y)$.

Since $f_x = 6xy - 6x$, $f_y = 3x^2 + 3y^2 - 6y$,

we have second partial derivatives:

$$f_{xx} = 6y - 6, \quad f_{xy} = f_{yx} = 6x, \quad f_{yy} = 6y - 6.$$

So

$$\begin{aligned} D(x, y) &= f_{xx} f_{yy} - (f_{xy})^2 \\ &= (6y-6)^2 - (6x)^2 \\ &= 36(y-1)^2 - 36x^2. \end{aligned}$$

(3) Classify each critical point

For $(0, 0)$,

$$D(0,0) = 36(0-1)^2 - 36(0)^2 = 36 > 0$$

and $f_{xx}(0, 0) = 6(0) - 6 = -6 < 0$

$\Rightarrow$ $(0,0)$ is a local maximum.

For $(0, 2)$,

$$D(0, 2) = 36(2-1)^2 - 36(0)^2 = 36 > 0$$

and $f_{xx}(0, 2) = 6(2) - 6 = 6 > 0$

$\Rightarrow$ $(0, 2)$ is a local minimum

For $(1, 1)$,

$$D(1, 1) = 36(1-1)^2 - 36(1)^2 = -36 < 0$$

$\Rightarrow$ $(1, 1)$ is a saddle point

For $(-1, 1)$,

$$D(-1, 1) = 36(1-1)^2 - 36(-1)^2 = -36 < 0$$

$\Rightarrow$ $(-1, 1)$ is a saddle point.

In conclusion,

local maximum value $f(0, 0) = 2$,

local minimum value $f(0, 2) = -2$,

and $(1, 1)$, $(-1, 1)$ are saddle points of f.

Text-Ex 3: Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$

Ans: (1) Find all critical points

$$\nabla f(x, y) = \langle \underbrace{4x^3 - 4y}_{= f_x}, \underbrace{4y^3 - 4x}_{= f_y} \rangle$$

Setting $\nabla f(x, y) = \vec{0}$, we have

$$\begin{cases} 4x^3 - 4y = 0 & (1) \\ 4y^3 - 4x = 0 & (2) \end{cases}$$

From (1), $y = x^3$. Plug this into (2):

$$4\left((x^3)^3 - x\right) = 0$$

$$4(x^9 - x) = 0$$

$$x(x^8 - 1) = 0$$

$\Rightarrow$ $x = 0$ or $x = 1$ or $x = -1$.

If $x = 0$, then $y = x^3 = 0$, point $(0, 0)$

If $x = 1$, then $y = x^3 = 1$, point $(1, 1)$

If $x = -1$, then $y = x^3 = -1$, point $(-1, -1)$.

So we have three critical points $(0, 0)$, $(1, 1)$, $(-1, -1)$.

(2) Find the discriminant.

We have $f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$,

so $f_{xx} = 12x^2$, $f_{xy} = f_{yx} = -4$, $f_{yy} = 12y^2$.

Then $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = (12x^2)(12y^2) - (-4)^2$

$$= 144x^2y^2 - 16$$

(3) Classify each critical point.

For $(0, 0)$: $D(0, 0) = -16 < 0 \Rightarrow$ saddle point.

For $(1, 1)$: $D(1, 1) = 128 > 0$, $f_{xx}(1, 1) = 12 > 0$

$\Rightarrow$ local min

For $(-1, -1)$: $D(-1, -1) = 128 > 0$, $f_{xx}(-1, -1) = 12 > 0$

$\Rightarrow$ local min.

So local minimum values are $f(1,1) = -1$ and $f(-1,-1) = -1$

(f has local minimum values at $(1,1)$ and $(-1,-1)$, and the local minimum values are

$$f(1,1) = -1, \quad f(-1,-1) = -1. \;)$$

and f has a saddle point $(0,0)$.

3. Finding absolute max and min for $f(x,y)$ on a closed bounded region D.

Procedure: (1) Find all critical points in D. Take f at these points

(2) Find the max and min of $f(x,y)$ on the edge (on the boundary of D).

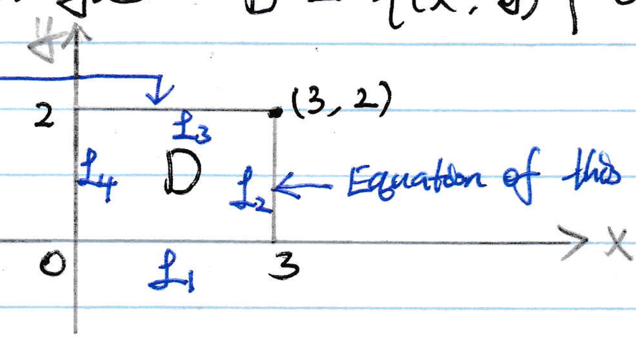
How this is done depends on f and the shape of D. The basic idea is to use the edge equation to change f into a function of one variable.

(3) Take the largest value and smallest value from step 1 and 2.

Text - Ex 7: Find the absolute maximum and minimum values of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

Equation of L_3

$y = 2,\ 0 \le x \le 3$.



Equation of this edge $x = 3$

$0 \le y \le 2$.

Ans: (1) Find all critical points and take f at these points.

$$\nabla f = \langle 2x - 2y \ , \ -2x + 2 \rangle$$

Set $\nabla f(x, y) = \vec{0}$: $\begin{cases} 2x - 2y = 0 & ① \\ -2x + 2 = 0 & ② \end{cases}$

$\Rightarrow \underset{\text{(from ②)}}{x = 1}, \quad \underset{\text{(from ①)}}{y = x = 1} \quad \Rightarrow$ A critical point $(1, 1)$.

Evaluate f : $f(1, 1) = 1$.

(2) Find the max and min of f on the boundary of D.

Note the boundary of D consists of four line segments L_1, L_2, L_3, L_4, see the drawing on the previous page.

On L_1, the equation of line segment is

$$y = 0 \ , \quad 0 \le x \le 3 .$$

Use this and we have

$$f(x, y) = x^2 - 2xy + 2y = x^2 - 2x(0) + 2(0)$$
$$= x^2 .$$

Since $0 \le x \le 3$, so the max of f on L_1 is attached at $(3, 0)$ and $f(3,0) = 9$; the min of f is attached at $(0, 0)$ and $f(0,0) = 0$.

On L_2, equation of the edge : $x = 3, \ 0 \le y \le 2$.
Plug this into $f(x, y) = x^2 - 2xy + 2y$.

$$f(x, y) = 3^2 - 2(3)y + 2y = 9 - 4y, \quad 0 \le y \le 2.$$

So on L_2, the max value of f is attached at $(3, 0)$ and $f(3,0) = 9$; the min value of f is attached at $(3, 2)$ and $f(3,2) = 1$.

On L_3, equation of the edge : $y = 2$, $0 \le x \le 3$.

So $\quad f(x, y) = x^2 - 4x + 4 = (x-2)^2$.

We see max is attached at $(0, 2)$ and $f(0,2) = 4$; min of f is attached at $(2, 2)$ and $f(2,2) = 0$.

On L_4, equation of the edge : $x = 0$, $0 \le y \le 2$.

So $\quad f(x, y) = 2y, \quad 0 \le y \le 2$.

We see max is attached at $(0, 2)$ and $f(0,2) = 4$; min of f is attached at $(0, 0)$ and $f(0,0) = 0$.

(3) All the values from step 1 and 2 :

1, 9, 0, 9, 1, 4, 0, 4, 0

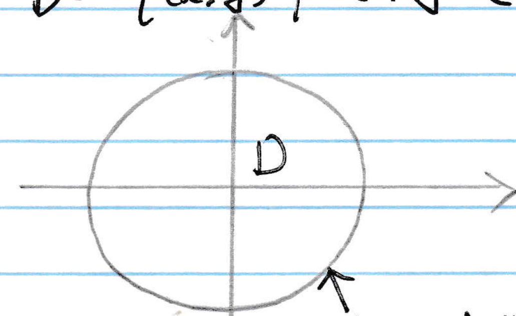
absolute max

absolute min

So the absolute max on D is attached at $(3, 0)$ and the max value is $f(3,0) = 9$; the absolute min on D is attached at both $(0,0)$ and $(2,2)$ and the min value is

$$f(0,0) = f(2,2) = 9$$

Example: Find the absolute maximum and minimum values of $f(x, y) = 2x^2 - 3y^2$ on $D = \{(x, y) \mid x^2 + y^2 \leq 4\}$



Equation of the edge: $x^2 + y^2 = 4$.

Ans: (1) Critical points

$\nabla f(x, y) = \langle 4x, -6y \rangle$

$$\begin{cases} 4x = 0 \\ -6y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}.$$

Critical point $(0, 0)$, $f(0, 0) = 0$.

(2) Max and min on the boundary

Edge equation: $x^2 + y^2 = 4$. $\Rightarrow y^2 = 4 - x^2$

So

$$f(x, y) = 2x^2 - 3y^2 = 2x^2 - 3(4 - x^2)$$

$$= 2x^2 - 12 + 3x^2 = 5x^2 - 12.$$

We know on the edge (circle), $-2 \leq x \leq 2$.

Therefore max of f on the boundary: $f(2, 0) = f(-2, 0) = 8$

min of f: $f(0, 2) = f(0, -2) = -12$.

(3) Absolute max $f(2, 0) = f(-2, 0) = 8$

Absolute min $f(0, 2) = f(0, -2) = -12$.