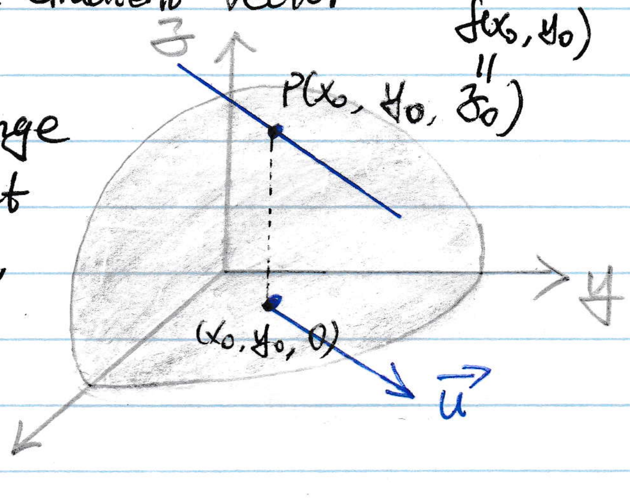
14.6 Directional Derivatives and the Gradient Vector

1. Recall that f_x means the change in f as x increases (the point (x, y) moves in the $\vec{i}$ direction), and f_y means the change in f as y increases (the point (x, y) moves in the $\vec{j}$ direction)



We may ask how f changes if we go in some other direction.

2. Definition of the directional derivative

For a unit vector $\vec{u} = \langle a, b \rangle$ or $\vec{u} = \langle a, b, c \rangle$,

2D case $f(x, y)$ — 3D case $f(x, y, z)$

the directional derivative of f in the direction of $\vec{u}$ is:

$$D_{\vec{u}} f = \underbrace{a f_x + b f_y}_{\text{2D case } f(x,y)} \quad \text{or} \quad \underbrace{a f_x + b f_y + c f_z}_{\text{3D case } f(x,y,z)}$$

A function in 2D or 3D.

Remark: The phrase "directional derivative in the direction of" is used even when the vector is not a unit vector but you must make it a unit vector before using the formula.

Text-Ex 2: Find the directional derivative $D_{\vec{u}} f(x, y)$ for

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\vec{u} = \langle \cos(\theta), \sin(\theta) \rangle$ with $\theta = \frac{\pi}{6}$. What is $D_{\vec{u}} f(1, 2)$?

Ans: $\vec{u} = \langle \cos(\frac{\pi}{6}), \sin(\frac{\pi}{6}) \rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

We also have the first partial derivatives of f:

$$f_x(x, y) = 3x^2 - 3y, \quad f_y(x, y) = -3x + 8y.$$

therefore

$$\begin{aligned} D_{\vec{u}} f(x, y) &= a f_x(x, y) + b f_y(x, y) \\ &= \frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(-3x + 8y) \\ &= \frac{3\sqrt{3}}{2} x^2 - \frac{3}{2} x + \left(4 - \frac{3\sqrt{3}}{2}\right) y \end{aligned}$$

To find $D_{\vec{u}} f(1, 2)$, we plug in $(x, y) = (1, 2)$:

$$\begin{aligned} D_{\vec{u}} f(1, 2) &= \frac{3\sqrt{3}}{2}(1)^2 - \frac{3}{2}(1) + \left(4 - \frac{3\sqrt{3}}{2}\right)(2) \\ &= \frac{13 - 3\sqrt{3}}{2}. \end{aligned}$$

Text-Ex 5(b): Find the directional derivative of

$f(x, y, z) = x \sin(yz)$ at $(1, 3, 0)$ in the

direction of $\vec{v} = \vec{i} + 2\vec{j} - \vec{k} = \langle 1, 2, -1 \rangle$

Not a unit vector, we need to find the normal vector first.

Ans: The unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, 2, -1 \rangle}{\sqrt{1^2 + 2^2 + (-1)^2}} = \frac{\langle 1, 2, -1 \rangle}{\sqrt{6}}$$

$$\vec{u} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$$

So by the definition,

$$D_{\vec{u}} f(x, y) = a f_x(x, y, z) + b f_y(x, y, z) + c f_z(x, y, z)$$

$$= \frac{1}{\sqrt{6}} \left(\sin(yz) \right) + \frac{2}{\sqrt{6}} \left(x \cos(yz)(z) \right) - \frac{1}{\sqrt{6}} \left(x \cos(yz)(y) \right)$$

$$\Rightarrow D_{\vec{u}} f(x, y) = \frac{1}{\sqrt{6}} \sin(yz) + \frac{2}{\sqrt{6}} xz \cos(yz) - \frac{1}{\sqrt{6}} xy \cos(yz).$$

Plug in $(x, y, z) = (1, 3, 0)$:

$$D_{\vec{u}}(1, 3, 0) = \frac{1}{\sqrt{6}} \sin(0) + \frac{2}{\sqrt{6}}(0)\cos(0) - \frac{1}{\sqrt{6}}(1)(3)\cos(0)$$

$$\Rightarrow D_{\vec{u}}(1, 3, 0) = -\frac{3}{\sqrt{6}}$$

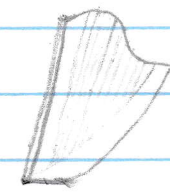
3. (Not Needed) Rigorous Definition of the directional derivative using limits:

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

where $\vec{u} = \langle a, b \rangle$ is a unit vector.

4. Definition of the gradient

<u>Notation</u>: Grad f or ∇f ("∇" is pronounced "nabla" and comes from the Hellenistic Greek word for a Phoenician harp)



Definition:

Vector-valued function

$$\nabla f = \underbrace{\langle f_x, f_y \rangle}_{f_x \vec{i} + f_y \vec{j}} \quad \text{or} \quad \underbrace{\langle f_x, f_y, f_z \rangle}_{f_x \vec{i} + f_y \vec{j} + f_z \vec{k}}$$

2D Case $f(x, y)$ 3D Case $f(x, y, z)$

<u>Text-Ex 3</u>: For $f(x, y) = \sin(x) + e^{xy}$, we have

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos(x) + e^{xy}(y), e^{xy}(x) \rangle$$

$$\nabla f(x, y) = \langle \cos(x) + ye^{xy}, xe^{xy} \rangle .$$

If we want to know $\nabla f(0, +1)$, the just plug in $(x, y) = (0, +1)$:

$$\nabla f(0, +1) = \langle \cos(0) + (+1)e^{(0)}, (0)e^{(0)} \rangle$$

$$= \langle 2, 0 \rangle .$$

5. Basic properties

(1) Observe that since $|\vec{u}| = 1$, we have:

b/c $D_{\vec{u}} f = a f_x + b f_y$, $\vec{u} = \langle a, b \rangle$, $\nabla f = \langle f_x, f_y \rangle$

$$D_{\vec{u}} f = \vec{u} \cdot \nabla f = \underbrace{|\vec{u}|}_{=1} \, |\nabla f| \cos(\theta) = |\nabla f| \cos(\theta)$$

where θ is the angle between $\vec{u}$ and ∇f.

Example: Find the gradient of function $f(x, y) = x^2 y$. Also find the directional derivative in the direction of $\vec{v} = 3\vec{i} + 4\vec{j}$.

Ans: $$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$
$$= \langle 2xy, x^2 \rangle$$

We normalize $\vec{v}$ to find the unit vector

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 3, 4 \rangle}{\sqrt{3^2 + 4^2}} = \frac{\langle 3, 4 \rangle}{5} = \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle.$$

Then we use the formula $D_{\vec{u}} f = \vec{u} \cdot \nabla f$ to compute the directional derivative:

$$D_{\vec{u}} f(x, y) = \vec{u} \cdot \nabla f(x, y)$$
$$= \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle \cdot \langle 2xy, x^2 \rangle$$
$$= \tfrac{3}{5}(2xy) + \tfrac{4}{5} x^2$$

$$\Rightarrow \quad D_{\vec{u}} f(x, y) = \tfrac{6}{5} xy + \tfrac{4}{5} x^2.$$

It follows from $D_{\vec{u}} f = |\nabla f| \cos(\theta)$ that $D_{\vec{u}} f$ is largest when $\theta = 0$ in which case $\vec{u}$ points in the same direction as ∇f and $D_{\vec{u}} f = |\nabla f|$. So we have properties:

- (2) First this means that ∇f points in the direction of maximum instantaneous increase of f
- (3) Second this means that the largest possible $D_{\vec{u}} f$ is in fact $|\nabla f|$

(2) & (3) together

- (4) Different $\vec{u}$ give different values for $D_{\vec{u}} f$. The largest value is when $\vec{u} = \left(\frac{\nabla f}{|\nabla f|}\right)$ and that largest value is $|\nabla f|$. (Note that the smallest value of $D_{\vec{u}} f$ is $-|\nabla f|$ when $\vec{u} = -\frac{\nabla f}{|\nabla f|}$)

<u>Example</u>: If the temperature at (x, y) is $f(x, y) = x^2 y$ and a bug is at $(1, 2)$, in which direction does it detect the greatest increase in temperature and what is that increase?

<u>Ans</u>:
$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$
$$= \langle 2xy, x^2 \rangle$$
$$\Rightarrow \nabla f(1, 2) = \langle 4, 1 \rangle.$$

Therefore the direction that gives the greatest increase in temperature is

$$\vec{u} = \frac{\nabla f(1,2)}{|\nabla f(1,2)|} = \frac{\langle 4, 1\rangle}{\sqrt{4^2+1^2}} = \left\langle \frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \right\rangle$$

unit vector parallel to $\nabla f(1,2)$.

The increase of temperature in the direction $\vec{u}$ is

$$D_{\vec{u}} f(1,2) = |\nabla f(1,2)| = \sqrt{17}.$$

Only true when $\vec{u} = \frac{\nabla f}{|\nabla f|}$, in general $D_{\vec{u}} f = \vec{u} \cdot \nabla f$.

6. Normal / Perpendicular properties

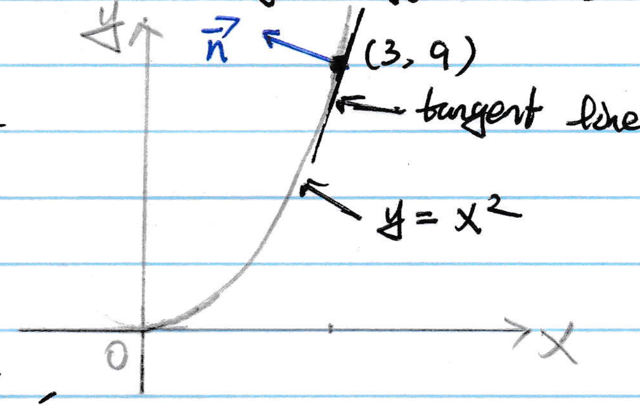
(1) $\nabla f(x, y)$ is normal to the level curve of $f(x, y)$ at (x, y).

In other words, for a level curve $f(x, y) = k$, if (x_0, y_0) is on this level curve, i.e $f(x_0, y_0) = k$, then $\nabla f(x_0, y_0)$ is normal to the level curve $f(x, y) = k$ at (x_0, y_0).

<u>Example</u>: Find a vector normal to $y = x^2$ at $(3, 9)$.

<u>Ans</u>: Set $f(x, y) = y - x^2$, then we see

that $(3, 9)$ is on the curve $f(x, y) = 0$.



So a vector $\perp$ to the curve is $\nabla f(3, 9)$.

Since $\nabla f(x, y) = \langle f_x, f_y \rangle$
$= \langle -2x, 1 \rangle$,

we have $\vec{n} = \nabla f(3, 9) = \boxed{\langle -6, 1 \rangle}$.

(One comment is that one can easily verify that the tangent line $\parallel \langle 1, \underline{2x} \rangle = \langle 1, 6 \rangle$

derivative of x^2

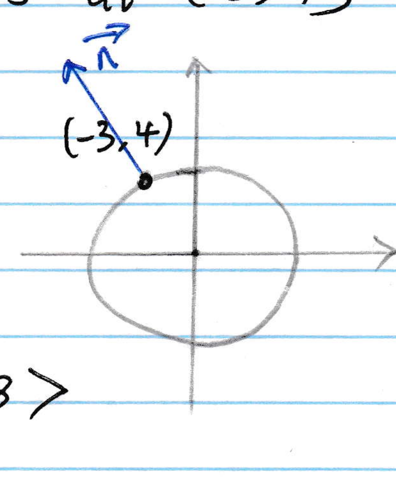
and $\vec{n} = \langle -6, 1 \rangle \perp \langle 1, 6 \rangle$.)

Example: Find a vector $\perp$ to the curve

$$x^2 + y^2 = 25 \quad \text{at} \quad (-3, 4).$$

Ans: Let $f(x, y) = x^2 + y^2$.

Then a normal vector of $f(x, y) = 25$ at $(-3, 4)$ is $\nabla f(-3, 4)$.



Since $\nabla f(x, y) = \langle f_x, f_y \rangle$
$= \langle 2x, 2y \rangle$,

we have $\vec{n} = \nabla f(-3, 4) = \langle -6, 8 \rangle$

(2) $\nabla f(x, y, z)$ is normal to the level surface of $f(x, y, z)$ at (x, y, z).

Example: Find a vector $\perp$ to the surface

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \quad \text{at } (-2, 1, -3).$$

Ans: Let $f(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$.

then

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

$$= \langle \tfrac{1}{2}x, 2y, \tfrac{2}{9}z \rangle.$$

So a normal vector to the surface at $(-2, 1, -3)$ is

$$\nabla f(-2, 1, -3) = \langle -1, 2, -\tfrac{2}{3} \rangle.$$

(3) Normal line of $f(x, y, z) = k$ at (x_0, y_0, z_0).

is a line passing through (x_0, y_0, z_0) and $\parallel \nabla f(x_0, y_0, z_0)$.

(It follows from Sec 12.5, equations of lines)

So the symmetric equations of the normal line are

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}$$

and the parametric equations are

$$x = x_0 + f_x(x_0, y_0, z_0)\, t, \quad y = y_0 + f_y(x_0, y_0, z_0)\, t, \quad z = z_0 + f_z(x_0, y_0, z_0)\, t$$

(For 2D, function $f(x, y)$, the above theory works, just no z component)

(14) Tangent plane of $f(x, y, z) = k$ at (x_0, y_0, z_0) is a plane passing through (x_0, y_0, z_0) and $\perp$ $\nabla f(x_0, y_0, z_0)$

Equation $\quad f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$

or $\quad \nabla f(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$.

Text-Ex 8 : Find the equation of the tangent plane and normal line of the surface

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \quad \text{at} \quad (-2, 1, -3).$$

Ans : Let $f(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$, we have already computed in the previous example that

$$\nabla f(\underset{-2}{\underset{\parallel}{x_0}}, \underset{1}{\underset{\parallel}{y_0}}, \underset{-3}{\underset{\parallel}{z_0}}) = \langle -1, 2, -\tfrac{2}{3} \rangle.$$

So the equation of the tangent plane is

$$(-1)(x - \overset{-2}{\overset{\parallel}{x_0}}) + (2)(y - \overset{1}{\overset{\parallel}{y_0}}) + (-\tfrac{2}{3})(z - \overset{-3}{\overset{\parallel}{z_0}}) = 0$$

$$\boxed{-x + 2y - \tfrac{2}{3}z - 6 = 0}.$$

The normal line has symmetric equations :

$$\frac{x - (-2)}{-1} = \frac{y - (1)}{2} = \frac{z - (-3)}{-\frac{2}{3}}$$

$$\boxed{\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\frac{2}{3}}}$$