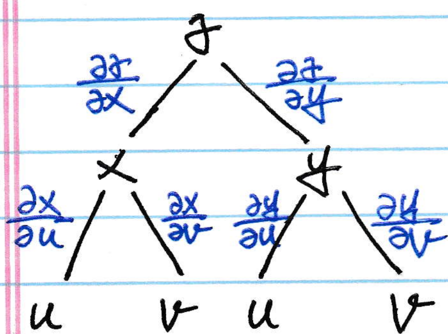
14.5 The Chain Rule

1. When do we use chain rule? What is a tree diagram?

Example: (1) $z = x^3y + y^2$, $x = u\sin(v)$, $y = v\cos(u)$

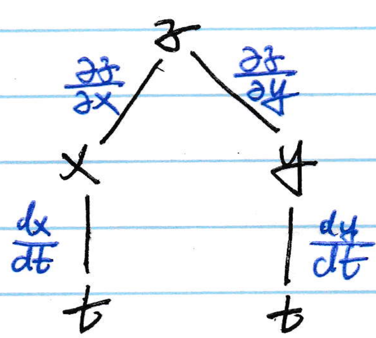
Then we can think z as a function of u and v.

So $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$ make sense. (Of course one can also treat z as a function of x and y, so $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ make sense)



← tree diagram

(2) $z = x^2y + 3xy^4$, $x = \sin(2t)$, $y = \cos(t)$



tree diagram

z is a function of x and y, so $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ make sense.

z is also a function of t, so $\frac{dz}{dt}$ makes sense

partial derivatives ordinary derivative

Comments: Why do we have $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ but $\frac{dz}{dt}$?

Because if we treat z as a function of x, y, it is a function of $\boxed{\text{two}}$ variables, while if we treat z as a function of t,

z is a function of one variable t.

2. How do we apply the chain rule?

Step 1: Draw a tree diagram.
Step 2: On each brach put either a "d" or a "∂" depending on whether it's a regular (ordinary) derivative (one variable) or a partial derivative.

dependent variable

Step 3: Find all routes from the top of the tree to the variable we're taking the derivative with respect to.

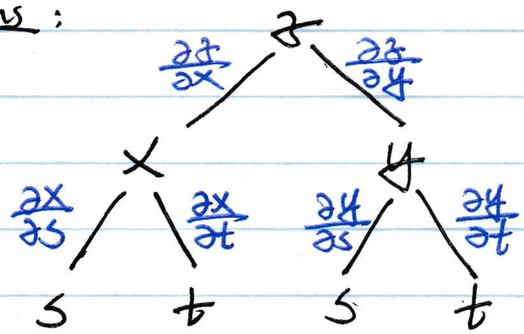
Step 4: Along each path find the derivatives and multiply.

Step 5: Add the paths.

Step 6: Substitute in for the final variable(s).

Text-Ex 3: $z = e^x \sin(y)$, $x = st^2$ and $y = s^2t$.
Find $\partial z/\partial s$ and $\partial z/\partial t$.

Ans:



$\leftarrow$ We do step 1 and step 2.

Steps 3, 4, 5

Two routes from z to s.

$$\frac{\partial z}{\partial x} = e^x \sin(y) \quad , \quad \frac{\partial x}{\partial s} = t^2$$

$$\frac{\partial z}{\partial y} = e^x \cos(y) \quad , \quad \frac{\partial y}{\partial s} = 2st$$

So $$\frac{\partial z}{\partial s} = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial s}\right)$$

$$= \left(e^x \sin(y)\right)\left(t^2\right) + \left(e^x \cos(y)\right)\left(2st\right)$$

Step 6

plug in $x = st^2$, $y = s^2t$.

$$= e^{st^2} \sin(s^2t)\, t^2 + e^{st^2} \cos(s^2t)(2st)$$

$$\left(= t^2 e^{st^2} \sin(s^2t) + 2st\, e^{st^2} \cos(s^2t) \right)$$

Redo steps 3-6 for $\frac{\partial z}{\partial t}$. Two routes.

$\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ already computed

$$\frac{\partial x}{\partial t} = 2st \quad , \quad \frac{\partial y}{\partial t} = s^2$$

So $$\frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right)$$

$$= \left(e^x \sin(y)\right)\left(2st\right) + \left(e^x \cos(y)\right)\left(s^2\right)$$

Step 6

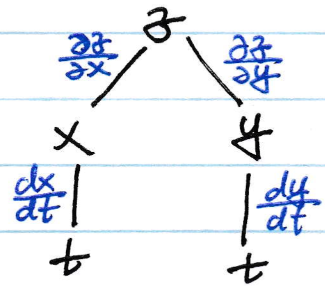
plug in $x = st^2$, $y = s^2t$

$$= e^{st^2} \sin(s^2t)\,(2st) + e^{st^2} \cos(s^2t)\, s^2$$

$$\left(= 2st\, e^{st^2} \sin(s^2t) + s^2 e^{st^2} \cos(s^2t) \right)$$

Text-Ex 1 For $z = x^2 y + 3xy^4$ where $x = \sin(2t)$ and $y = \cos(t)$. Find $\frac{dz}{dt}$ when $t = 0$.

Ans:



tree diagram

$$\frac{\partial z}{\partial x} = 2xy + 3y^4$$

$$\frac{\partial z}{\partial y} = x^2 + 3x\,(4y^3) = x^2 + 12xy^3$$

$$\frac{dx}{dt} = \cos(2t)\,(2) = 2\cos(2t)$$

$$\frac{dy}{dt} = -\sin(t)$$

So

$$\frac{dz}{dt} = \left(\frac{\partial z}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{dy}{dt}\right)$$

$$= \left(2xy + 3y^4\right)\left(2\cos(2t)\right) + \left(x^3 + 12xy^3\right)\left(-\sin(t)\right)$$

Plug in $x = \sin(2t)$, $y = \cos(t)$

$$= \left(2\sin(2t)\cos(t) + 3\cos^4(t)\right)\left(2\cos(2t)\right)$$

$$+ \left(\sin^3(2t) + 12\sin(2t)\cos^3(t)\right)\left(-\sin(t)\right)$$

So when $t = 0$,

$$\left.\frac{dz}{dt}\right|_{t=0} = \left(2(0)(1) + 3(1)^4\right)\left(2(1)\right)$$

$$+ \left(0^3 + 12(0)(1)^3\right)(-0)$$

$$= (3)(2) + (0)(0) = 6$$

Remark: In fact, we do not need to plug in $x = \sin(2t)$ and $y = \cos(t)$ when finding the expression of $\frac{dz}{dt}$.
When $t = 0$, we have $x = \sin(2t) = 0$, $y = \cos(t) = 1$.

Plug $t = 0$, $x = 0$, $y = 1$ into the expression

$$\frac{dz}{dt} = (2xy + 3y^4)(2\cos(2t)) + (x^3 + 12xy^3)(-\sin(t)),$$

and we get

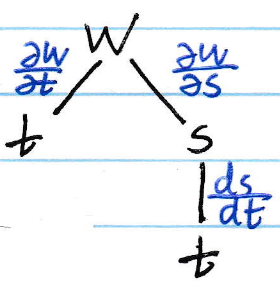
$$\left.\frac{dz}{dt}\right|_{t=0} = (2(0)(1) + 3(1)^4)(2(1)) + (0^3 + 12(0)(1)^3)(-0)$$

$$= (3)(2) + (0)(0) = 6.$$

(3) Sometimes tree branches may not be the same length.

Example: For $w = t^2 + \frac{1}{s}$ and $s = t^3 + t$, find $\frac{dw}{dt}$.

Ans: Notice we are asked to compute $\frac{dw}{dt}$, not $\frac{\partial w}{\partial t}$. What's the difference? We are treating w as a function of t (only one variable).



$$\frac{\partial w}{\partial t} = 2t, \quad \frac{\partial w}{\partial s} = -\frac{1}{s^2}$$

$$\frac{ds}{dt} = 3t^2 + 1.$$

So

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \left(\frac{\partial w}{\partial s}\right)\left(\frac{ds}{dt}\right)$$

$$= (2t) + \left(-\frac{1}{s^2}\right)(3t^2+1)$$

$$= 2t + \left(-\frac{1}{(t^3+t)^2}\right)(3t^2+1)$$

$$= 2t - \frac{3t^2+1}{(t^3+t)^2}.$$

3. Application: finding rates.

Text-Ex 2: For an ideal gas, we have

pressure (in kilopascals) $\rightarrow$ $$P = 8.31\,\frac{T}{V}$$ $\leftarrow$ temperature (in kelvins), $\leftarrow$ volume (in liters)

Find the rate at which the pressure is changing when $T = 300\,K$ and increasing at a rate of $0.1\,K/s$, and $V = 100\,L$ and increasing at a rate of $0.2\,L/s$.

Ans:

$$\begin{tikzcd}[column sep=small]
 & P \arrow[dl, "\frac{\partial P}{\partial T}"'] \arrow[dr, "\frac{\partial P}{\partial V}"] & \\
T \arrow[d, "\frac{dT}{dt}"'] & & V \arrow[d, "\frac{dV}{dt}"] \\
t & & t
\end{tikzcd}$$

time (in seconds)

$$\frac{dP}{dt} = \left(\frac{\partial P}{\partial T}\right)\left(\frac{dT}{dt}\right) + \left(\frac{\partial P}{\partial V}\right)\left(\frac{dV}{dt}\right)$$

$$\frac{\partial P}{\partial T} = \frac{8.31}{V}, \quad \frac{\partial P}{\partial V} = -8.31\,\frac{T}{V^2}.$$

Plug in $T = 300\,(K)$ and $V = 100\,(L)$;

$$\left.\frac{\partial P}{\partial T}\right|_{T=300,\,V=100} = \frac{8.31}{100}, \quad \left.\frac{\partial P}{\partial V}\right|_{T=300,\,V=100} = (-8.31)\frac{300}{100^2}.$$

From the description:

$$\frac{dT}{dt} = 0.1\,(K/s), \quad \frac{dV}{dt} = 0.2\,(L/s).$$

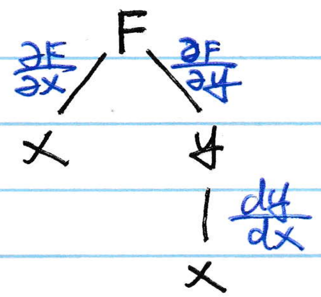
So $$\frac{dP}{dt} = \left(\frac{8.31}{100}\right)(0.1) + (-8.31)\left(\frac{300}{100^2}\right)(0.2)$$

$$= -0.04155\,(kPa/s)$$

4. Implicit function, implicit differentiation (We do not need this much)

Text-Ex 8: y is a function of x and implicitly defined by $x^3 + y^3 = 6xy$. Find $\frac{dy}{dx}$.

Ans: Write $F(x, y) = x^3 + y^3 - 6xy = 0$.



Treat F as a function of x.

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \left(\frac{\partial F}{\partial y}\right)\left(\frac{dy}{dx}\right).$$

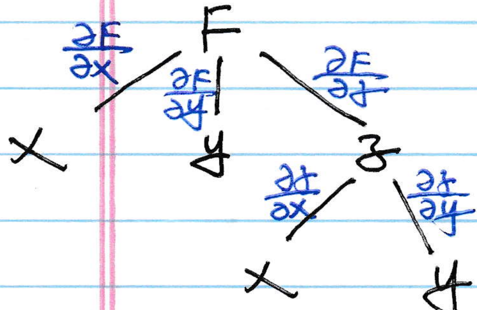
$\frac{dF}{dx} = 0$ b/c $F \equiv 0$

So: $$\frac{\partial F}{\partial x} + \left(\frac{\partial F}{\partial y}\right)\left(\frac{dy}{dx}\right) = 0$$

Formula [6] in the book $\rightarrow$ $$\Rightarrow \frac{dy}{dx} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial y}\right)} = -\frac{3x^2 - 6y}{3y^2 - 6x} = \boxed{\frac{2y - x^2}{y^2 - 2x}}$$

Text-Ex 9: z is a function of x and y, and implicitly defined by $x^3 + y^3 + z^3 + 6xyz = 1$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

Ans: Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1 = 0$.



We treat F as a function of x and y. To make it clear, we denote this as

$$\tilde{F}(x, y) = F(x, y, z)$$

a function of x, y: $z(x, y)$

Then $\frac{\partial \hat{F}}{\partial x}$ would be:

$$\frac{\partial \hat{F}}{\partial x} = \frac{\partial F}{\partial x} + \left(\frac{\partial F}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right)$$

These two are different. This is why we define a new function $\hat{F}$ which is F as a function of x and y. Here F is thought as a function of x, y and z.

Since $\hat{F}(x,y) = F(x,y,z) = 0$, we have $\frac{\partial \hat{F}}{\partial x} = 0$

$$\Rightarrow \quad 0 = \frac{\partial F}{\partial x} + \left(\frac{\partial F}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right) \Rightarrow \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$$

Therefore

$$\frac{\partial z}{\partial x} = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy}$$

Formula $\boxed{7}$ in the book

Similarly, consider $\frac{\partial \hat{F}}{\partial y}$ and we have

$$0 = \frac{\partial \hat{F}}{\partial y} = \frac{\partial F}{\partial y} + \left(\frac{\partial F}{\partial z}\right)\left(\frac{\partial z}{\partial y}\right) \Rightarrow \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

Therefore $\frac{\partial z}{\partial y} = -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}$.