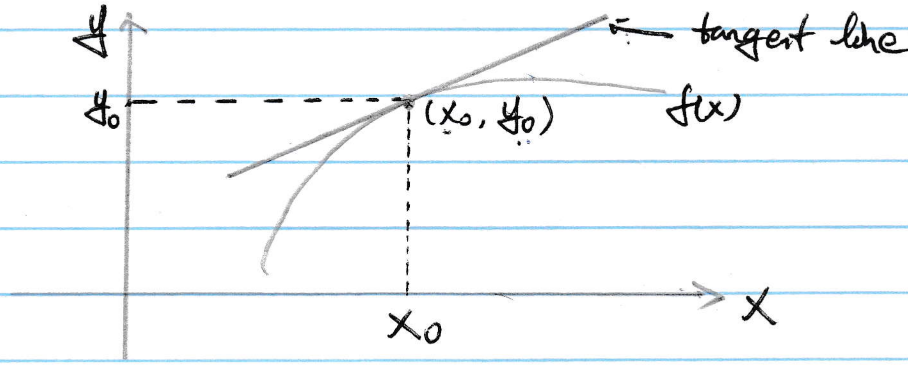
14.4 Tangent Planes and Linear Approximation

1. Recall the tangent line and linear approximation in 1D:



Equation of the tangent line at (x_0, y_0)

$$y = \underbrace{y_0}_{= f(x_0)} + \underbrace{f'(x_0)}_{= \text{slope of the tangent line}} (x - x_0).$$

This gives a linear approximation of $f(x)$:

$$f(x) \approx \underbrace{y_0}_{f(x_0)} + f'(x_0)(x - x_0).$$

Notice that this is the beginning of the Taylor Series:

$$f(x) = \underbrace{y_0}_{f(x_0)} + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots$$

2. How we do approximation in 2D:

We discussed the partial derivatives which give us the changing rates of function values in x or y direction.

$$f_x(x_0, y_0) \approx \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$f_y(x_0, y_0) \approx \frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}.$$

Therefore we have

$$f(x_0+\Delta x, y_0) \approx f(x_0, y_0) + (\Delta x)\, f_x(x_0, y_0)$$

$$f(x_0, y_0+\Delta y) \approx f(x_0, y_0) + (\Delta y)\, f_y(x_0, y_0).$$

So if we change in both x- and y-directions,

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0, y_0) + (\Delta x) f_x(x_0, y_0) + (\Delta y) f_y(x_0, y_0).$$

This can also be written as

$$\boxed{f(x,y) \approx f(x_0, y_0) + f_x(x_0, y_0)\,(x-x_0) + f_y(x_0, y_0)\,(y-y_0)}$$

Example: Approximate $\sqrt{3.02^2 + 6.95}$.

Ans: Notice that it is close to $\sqrt{3^2+7} = \sqrt{16} = 4$.

So we define $f(x,y) = \sqrt{x^2+y}$.

Then at $(x_0, y_0) = (3, 7)$, use

$$f(x,y) \approx f(x_0, y_0) + f_x(x_0, y_0)\,(x-x_0) + f_y(x_0, y_0)\,(y-y_0)$$

for $(x,y) = (3.02, 6.95)$.

We need to compute $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$.

$$f_x(x,y) = \frac{1}{2}(x^2+y)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+y}}$$

$$f_y(x,y) = \frac{1}{2}(x^2+y)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x^2+y}}.$$

Hence $f_x(x_0, y_0) = f_x(3, 7) = \dfrac{3}{\sqrt{3^2+7}} = \dfrac{3}{4}$

$f_y(x_0, y_0) = f_y(3, 7) = \dfrac{1}{2\sqrt{3^2+7}} = \dfrac{1}{8}$.

So:

$$\underbrace{f(3.02, 6.95)}_{=\sqrt{3.02^2+6.95}} \approx \underbrace{f(x_0, y_0)}_{=\sqrt{3^2+7}=4} + \frac{3}{4}(3.02-3) + \frac{1}{8}(6.95-7)$$

$$= 4 + \frac{3}{4}(0.02) + \frac{1}{8}(-0.05)$$

$$= 4.00875.$$

$\Rightarrow \sqrt{3.02^2+6.95} \approx 4.00875$.

In fact, the exact value computed by calculator is 4.00879....

3. Equation of the tangent plane

$$z = \underbrace{z_0}_{=f(x_0, y_0)} + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)$$

Text-Ex 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Ans: We are computing the tangent line of function $f(x, y) = 2x^2 + y^2$ at $(x_0, y_0) = (1, 1)$

Then $f_x(x, y) = 4x$, $f_y(x, y) = 2y$.

$$f_x(1,1) = 4 \quad , \quad f_y(1,1) = 2$$

So the tangent plane is

$$z = z_0 + 4(x - x_0) + 2(y - y_0)$$

$$z_0 = f(x_0, y_0) = 3 .$$

$$z = 3 + 4(x-1) + 2(y-1)$$

$$\boxed{z = 4x + 2y - 3} \leftarrow \text{Equation of the tangent plane}$$

4. We also have linear approximation for $f(x, y, z)$

function of three variables

$$f(x,y,z) \approx f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0)$$
$$+ f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

<u>Text-Exercise 21</u>: Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(3, 2, 6)$ and use it to approximate the number $\sqrt{3.02^2 + 1.97^2 + 5.99^2}$

<u>Ans</u>: $(x_0, y_0, z_0) = (3, 2, 6)$.

$$f_x(x,y,z) = \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$f_y(x,y,z) = \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2y) = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$f_z(x,y,z) = \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2z) = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$f(\underset{3}{\underset{\parallel}{x_0}}, \underset{2}{\underset{\parallel}{y_0}}, \underset{6}{\underset{\parallel}{z_0}}) = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$f_x(3,2,6) = \frac{3}{7}, \quad f_y(3,2,6) = \frac{2}{7}, \quad f_z(3,2,6) = \frac{6}{7}.$$

So the linear approximation is

$$f(x,y,z) \approx 7 + \frac{3}{7}(x-3) + \frac{2}{7}(y-2) + \frac{6}{7}(z-6).$$

To approximate $\sqrt{3.02^2 + 1.97^2 + 5.99^2}$, we have

$$\underbrace{\sqrt{3.02^2 + 1.97^2 + 5.99^2}}_{\substack{\| \\ f(3.02,\ 1.97,\ 5.99)}} \approx 7 + \frac{3}{7}(3.02-3) + \frac{2}{7}(1.97-2) + \frac{6}{7}(5.99-6)$$

$$= 7 + \frac{3}{7}(0.02) + \frac{2}{7}(-0.03) + \frac{6}{7}(-0.01)$$

$$= 7 - \frac{0.06}{7} = \boxed{\frac{2447}{350}}$$

5. Differentials (We don't use this)

For $z = f(x,y)$,

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy.$$

Similar to the linear approximation.