14.3 Partial Derivatives

1. Definition and Notation

(1) Consider $f(x, y)$, a function of two variables.
Then partial derivative of f with respect to x at (a, b) is

$$f_x(a,b) = \frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Two notations for partial derivatives

We also have partial derivative of f with respect to y at (a, b)

$$f_y(a,b) = \frac{\partial f}{\partial y}(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

(2) How do we compute partial derivatives?
Treat other variables as constants and take derivatives.

Text-Ex 1 If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Ans: For f_x (partial derivative w.r.t x), we treat y as a constant:

$$x^3 + x^2 y^3 - 2y^2$$

treat this as a constant

treat this as a constant

So taking derivative of x gives:

$$3x^2 + 2x\, y^3 - 0$$

This gives partial derivative of f w.r.t. x at a general point (x, y), i.e.

$$f_x(x, y) = 3x^2 + 2xy^3$$

To find out $f_x(2, 1)$, just plug $(x, y) = (2, 1)$ into the expression above

$$f_x(2, 1) = 3(2)^2 + 2(2)(1)^3 = 12 + 4 = 16.$$

Similarly for f_y (partial derivative w.r.t y), we treat x as a constant

$$x^3 + x^2 y^3 - 2y^2.$$

So taking derivative of y gives

$$0 + x^2(3y^2) - 4y \quad = 3x^2y^2 - 4y.$$

This tells us $\quad f_y(x, y) = 3x^2y^2 - 4y.$

To find $f_y(2, 1)$, plug in $(x, y) = (2, 1)$:

$$f_y(2, 1) = 3(2)^2(1)^2 - 4(1)$$

$$= 12 - 4 = 8.$$

Text-Ex 4 Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the function $\quad f(x, y) = \sin\left(\frac{x}{1+y}\right).$

Ans: To compute $f_x = \frac{\partial f}{\partial x}$, we use chain rule

$$f_x(x,y) = \cos\left(\frac{x}{1+y}\right) \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right)$$

treat y as a constant

$$= \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}.$$

For $f_y = \frac{\partial f}{\partial y}$, similarly

treat x as a constant

$$f_y(x,y) = \cos\left(\frac{x}{1+y}\right) \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right)$$

$$= \cos\left(\frac{x}{1+y}\right)\left(-\frac{x}{(1+y)^2}\right).$$

(3) Functions of more than two variables

Text-Ex 6: Find f_x, f_y and f_z for $f(x,y,z) = e^{xy} \ln(z)$.

Ans: To compute f_x, we treat both y and z as constants (all variables as constants except x).

$$f_x = \frac{\partial}{\partial x}\left(e^{xy} \ln(z)\right) = \ln(z)\, e^{xy} \frac{\partial}{\partial x}(xy)$$

$$= \ln(z)\, e^{xy}(y) = y \ln(z)\, e^{xy}.$$

For f_y, treat x and z as constants

$$f_y = e^{xy} \frac{\partial}{\partial y}(xy) \ln(z) = x \ln(z)\, e^{xy}.$$

And for f_z,

$$f_z = e^{xy} \cdot \frac{1}{z} = \frac{e^{xy}}{z}.$$

Example: Find f_x, f_y, f_z for $f(x,y,z) = \dfrac{xy - z}{x+y+z}$.

Ans: We need quotient rules in this example.

$$\left(\frac{d}{dx}\left(\frac{g(x)}{h(x)}\right) = \frac{g'(x)\,h(x) - g(x)\,h'(x)}{h^2(x)} \right)$$

So

$$f_x = \frac{\frac{\partial}{\partial x}(xy - z)\cdot(x+y+z) - (xy-z)\frac{\partial}{\partial x}(x+y+z)}{(x+y+z)^2}$$

$$= \frac{y(x+y+z) - (xy-z)\cdot(1)}{(x+y+z)^2}$$

$$= \frac{y^2 + yz + z}{(x+y+z)^2}$$

$$f_y = \frac{\frac{\partial}{\partial y}(xy - z)\cdot(x+y+z) - (xy-z)\frac{\partial}{\partial y}(x+y+z)}{(x+y+z)^2}$$

$$= \frac{x(x+y+z) - (xy-z)(1)}{(x+y+z)^2}$$

$$= \frac{x^2 + xz + z}{(x+y+z)^2}$$

$$f_z = \frac{\frac{\partial}{\partial z}(xy - z)\cdot(x+y+z) - (xy-z)\frac{\partial}{\partial z}(x+y+z)}{(x+y+z)^2}$$

$$= \frac{(-1)(x+y+z) - (xy-z)(1)}{(x+y+z)^2}$$

$$= \frac{-x - y - xy}{(x+y+z)^2}.$$

(4) For implicit functions.

<u>Text-Ex 5</u>: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

<u>Ans</u>: Take partial derivatives on both sides.

$$\frac{\partial}{\partial x}\left(x^3 + y^3 + z^3 + 6xyz\right) = \frac{\partial}{\partial x}(1)$$

$$3x^2 + 0 + 3z^2\frac{\partial}{\partial x}(z) + 6y\frac{\partial}{\partial x}(xz) = 0$$

(z is a function of x instead of a constant here!)

$$3x^2 + 3z^2\frac{\partial}{\partial x}(z) + 6y\left(x\frac{\partial}{\partial x}(z)\right) + 6y(z) = 0$$

$$\left(3z^2 + 6xy\right)\frac{\partial z}{\partial x} + \left(3x^2 + 6yz\right) = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$$

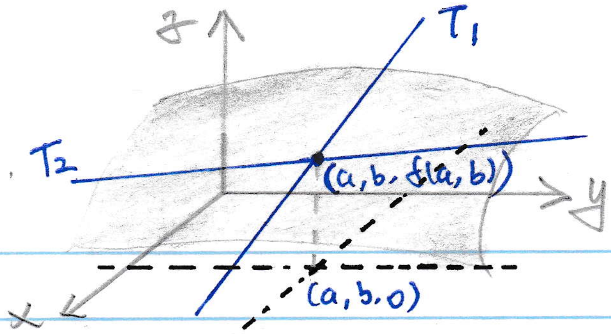
Also, $\frac{\partial}{\partial y}\left(x^3 + y^3 + z^3 + 6xyz\right) = \frac{\partial}{\partial y}(1)$

(z is a function of y instead of a constant)

$$0 + 3y^2 + 3z^2\frac{\partial z}{\partial y} + 6x\left(\left(\frac{\partial}{\partial y}(y)\right)z + y\frac{\partial}{\partial y}(z)\right) = 0$$

$$3y^2 + 3z^2\frac{\partial z}{\partial y} + 6xz + 6xy\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$



2. Methods of Visualization

(1) f_x gives the slope of the tangent line in the x-direction. f_y gives the slope of the tangent line in the y-direction (See Figure 1 in the textbook)

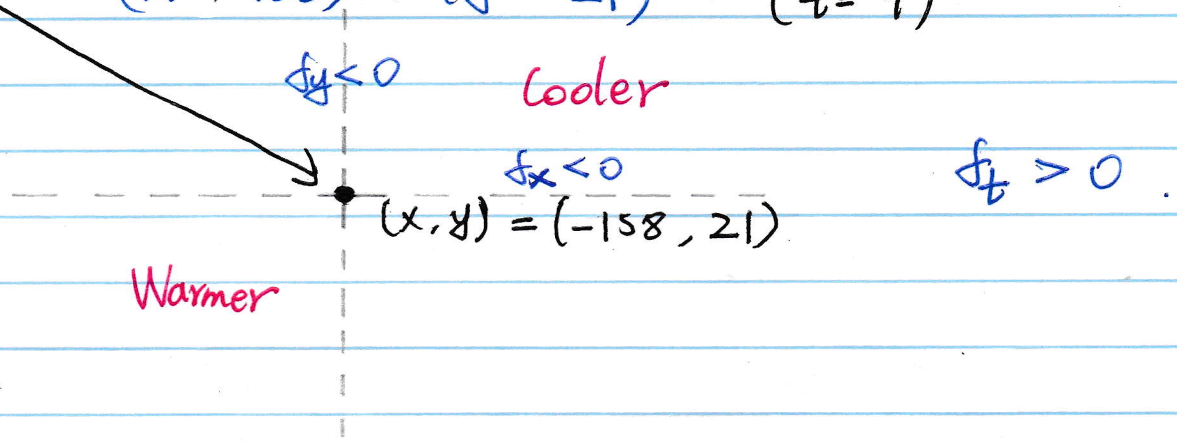
(2) If $f(x, y)$ gives the temperature of the plane at (x, y), then $f_x(x, y)$ gives the instantaneous temperature change of an object w.r.t. distance as it passes through (x, y) in the positive x-direction.

<u>Text-Exercise 1</u>: Temperature (in °C) $T = f(x, y, t)$

longitude, latitude, time

Honolulu: longitude 158°W, latitude 21°N, 9:00 AM

$(x = -158)$ $(y = 21)$ $(t = 9)$



3. Higher derivatives

Second partial derivatives:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y}$$

Mixed derivatives

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Text-Ex 1 : Find all second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2 .$$

Ans : $f_x(x, y) = 3x^2 + 2xy^3$

$f_y(x, y) = 3x^2y^2 - 4y$

(See the example on the first page)

$$f_{xx} = \frac{\partial}{\partial x}\big(\underbrace{3x^2 + 2xy^3}_{=f_x}\big) = 6x + 2y^3$$

$$f_{xy} = \frac{\partial}{\partial y}\big(\underbrace{3x^2 + 2xy^3}_{=f_x}\big) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}\big(\underbrace{3x^2y^2 - 4y}_{=f_y}\big) = 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y}\big(\underbrace{3x^2y^2 - 4y}_{=f_y}\big) = 6x^2y - 4 .$$

- Fact : Almost always $f_{xy} = f_{yx}$.

See the example above $f_{xy}(x, y) = 6xy^2 = f_{yx}(x, y)$.