14.1 Functions of Two Variables

1. Basic Definitions

(1) Function f of two variables (for example)

$$f(x, y) = x^2 + y^2$$

Here x, y are independent variables.

(Another notation $z = x^2 + y^2$)

with this notation, z is the dependent variable

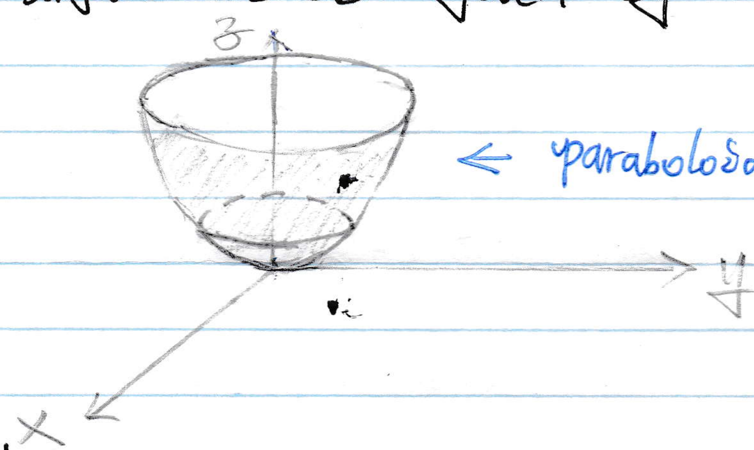
(2) Function f of three variables (for example)

$$f(x, y, z) = x + 2y + 3z$$

(Another notation $w = x + 2y + 3z$)

(3) Graph of functions.

For example the graph of $f(x, y) = x^2 + y^2$ is the surface in 3D given by $z = x^2 + y^2$.

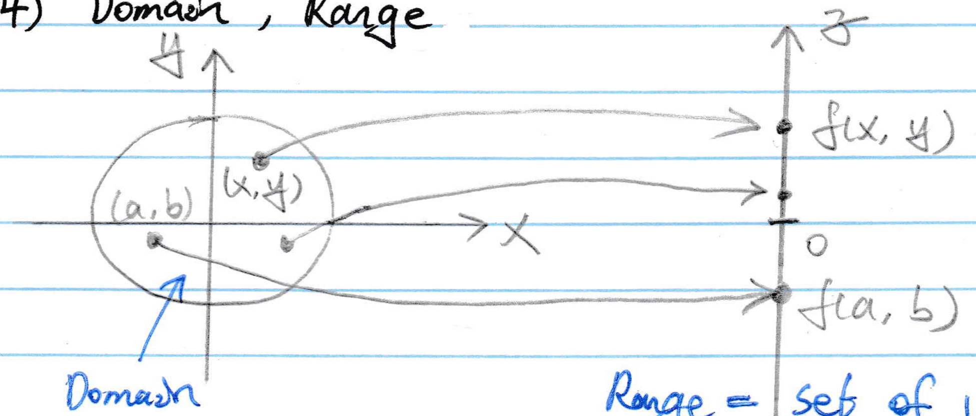


← paraboloid $z = x^2 + y^2$

(Note:
A surface in 3D may not be the graph of any functions
For example $x^2 + y^2 + z^2 = 1$)

Graph of $f(x, y, z)$, i.e. a function of three variables would be a hypersurface in 4D given by $w = f(x, y, z)$.

(4) Domain, Range



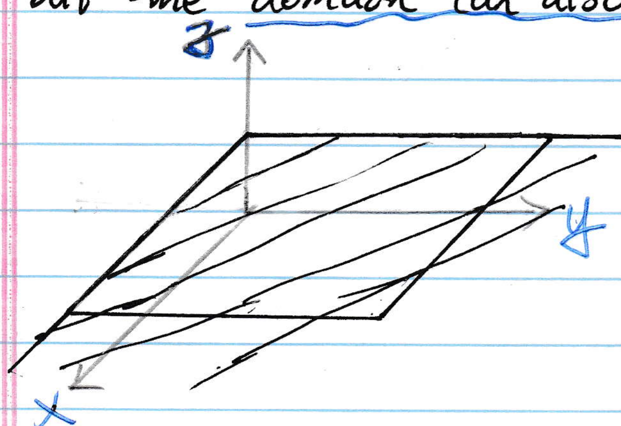
Example: $f(x, y) = \ln(1 - x^2 - y^2)$.

Ans: $1 - x^2 - y^2 > 0 \iff x^2 + y^2 < 1$

So the domain of the function f is

$$D = \{(x, y) \mid x^2 + y^2 < 1\}.$$

By convention, domain is the set of (x, y) such that the expression of f is well-defined (see example above), but the domain can also be given explicitly.



Example:

$f(x, y) = 3$, $x \geq 0$, $y \geq 0$

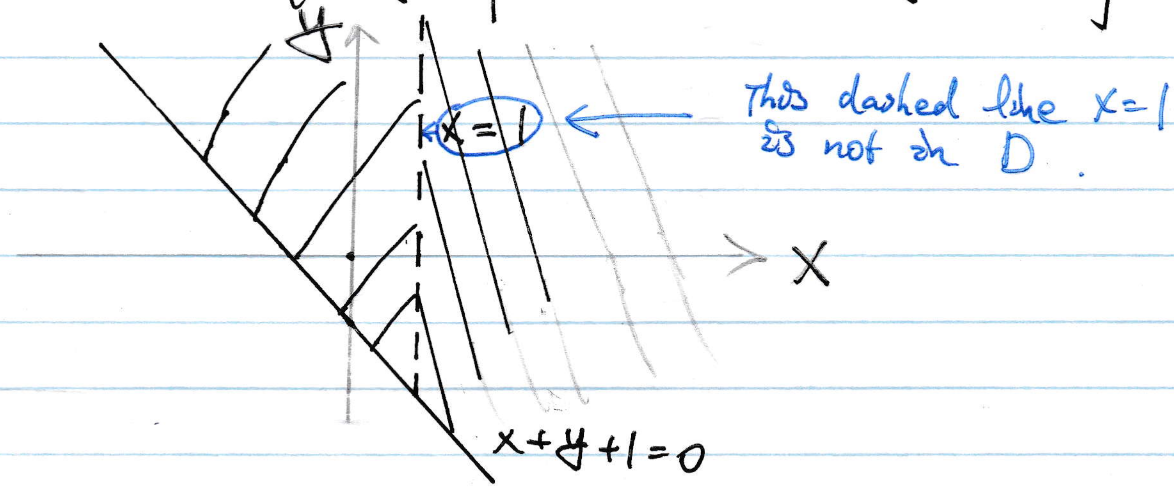
See the graph on the left

<u>Text-Ex 1</u> Evaluate $f(3,2)$ and sketch the domain

(a) $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$, (b) $f(x,y) = x\ln(y^2 - x)$

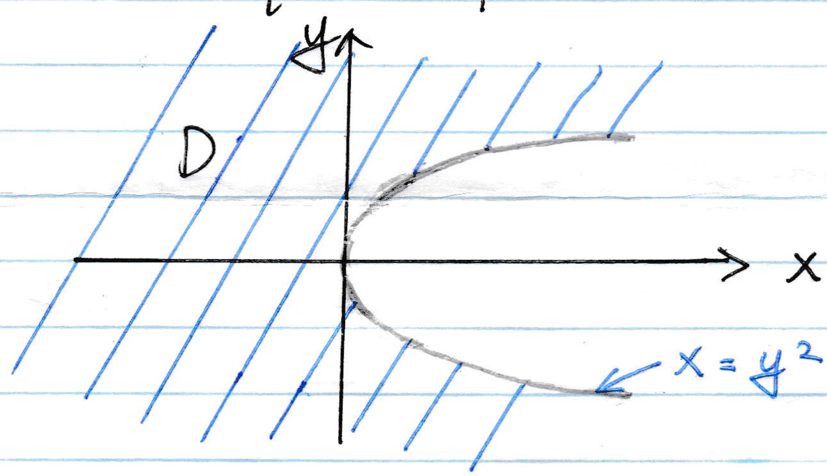
<u>Ans</u>: (a) $f(3,2) = \dfrac{\sqrt{3+2+1}}{3-1} = \dfrac{\sqrt{6}}{2}$

Domain $D = \{(x,y) \mid x-1 \neq 0 ,\ x+y+1 \geq 0\}$



(b) $f(3,2) = 3\ln(2^2 - 3) = 3\ln(1) = 0$

Domain $D = \{(x,y) \mid y^2 - x > 0\}$



Example: Find the range of $f(x,y) = x^2 + y^2$

Ans: The range is $\{z \mid z \geq 0\} = [0, +\infty)$.

One can find this by looking at the z-coordinates of the graph of f (graph plotted on the first page)

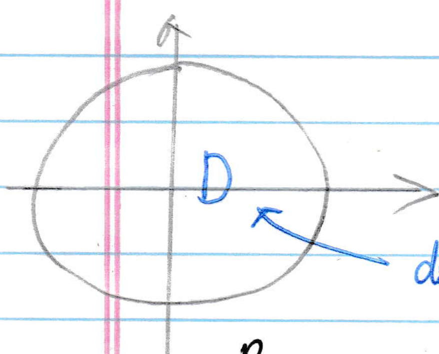
or by arguing that the minimum value of x^2+y^2 is 0 (when $x=y=0$), and no upper bound for x^2+y^2.

Text-Ex 4 Find the domain and range of

$$g(x,y) = \sqrt{9-x^2-y^2}$$

Ans: Domain

$$D = \{(x,y) \mid 9-x^2-y^2 \geq 0\}$$
$$= \{(x,y) \mid x^2+y^2 \leq 9\}$$



disk with center $(0,0)$ and radius $\sqrt{9} = 3$.

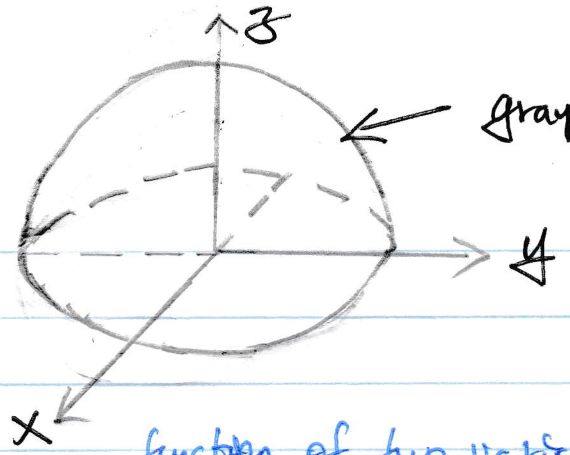
Range: possible values of $\sqrt{9-x^2-y^2}$?

We know $\underset{\text{trivial}}{0 \leq} x^2 + y^2 \underset{\text{domain}}{\leq} 9$

so $9-x^2-y^2 \in [0, 9]$, and hence

$\sqrt{9-x^2-y^2} \in [0, 3]$.

Therefore the range is $[0, 3]$.



graph of $z = \sqrt{9 - x^2 - y^2}$.

$(x^2 + y^2 + z^2 = 9)$

function of two variables   functions of three variables

2. Level Curves and Level Surfaces

Given $f(x, y)$. The level curves are the curves with equations $f(x, y) = k$, where k is a constant.

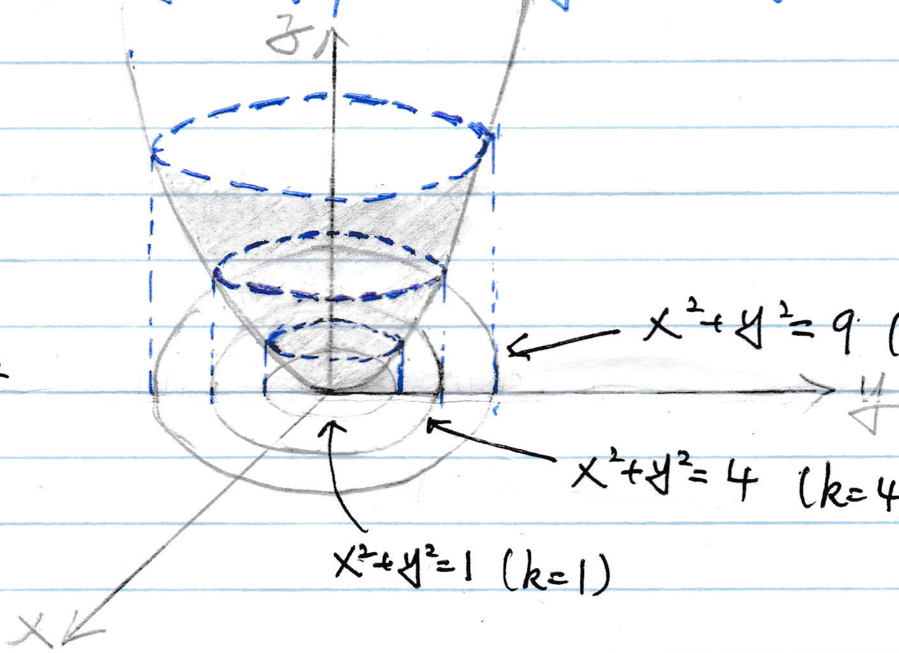
(See the visualization "Figure 11" in the textbook)

Given $f(x, y, z)$. The level surfaces are the surfaces with equations $f(x, y, z) = k$, where k is a constant

(see textbook Example 15 Figure 21 for a nice picture)

Example:

Level curves of $f(x, y) = x^2 + y^2$



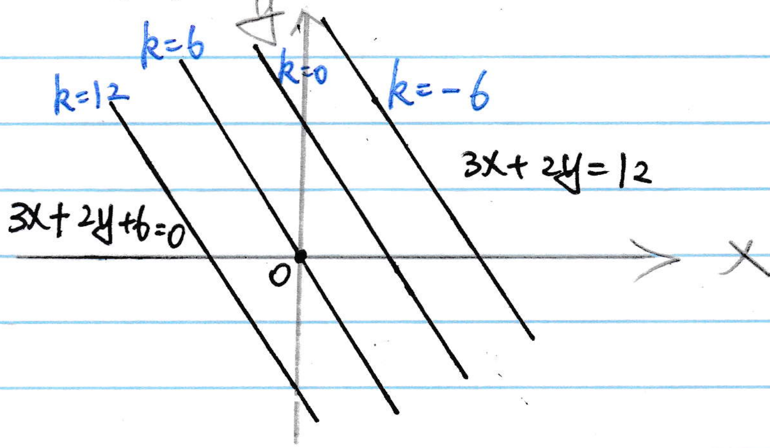
$x^2 + y^2 = 9$ $(k = 9)$

$x^2 + y^2 = 4$ $(k = 4)$

$x^2 + y^2 = 1$ $(k = 1)$

Text-Ex 10 Sketch the level curves of $f(x,y) = 6-3x-2y$ for $k = -6, 0, 6, 12$.

Ans:



level curve $6-3x-2y = k \iff 3x+2y = 6-k$

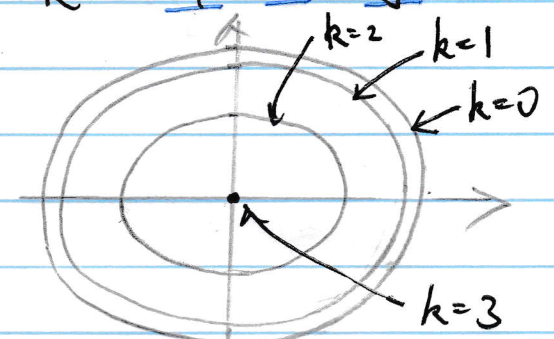
Text-Ex 11: Sketch the level curves of

$g(x,y) = \sqrt{9-x^2-y^2}$ for $k=0,1,2,3$

Ans: $k = \sqrt{9-x^2-y^2} \iff k^2 = 9-x^2-y^2$

$\iff x^2+y^2 = 9-k^2$

circle centered at $(0,0)$ with radius $\sqrt{9-k^2}$.



Text-Ex 15 Sketch the level surfaces of

$f(x,y,z) = x^2+y^2+z^2$.

Ans: $k = x^2+y^2+z^2$ sphere centered at $(0,0,0)$ with radius $\sqrt{k}$.