## 13.4 Motion in Space : Velocity and Acceleration

### 1. Velocity, Speed, Acceleration :

If $\vec{r}(t)$ gives position of an object then :

(1) $\vec{v}(t) = \vec{r}'(t)$ is the velocity (vector)

tangent vector in the direction of the tangent line

(2) $|\vec{v}(t)| = |\vec{r}'(t)|$ is the speed

(3) $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$ is the acceleration (vector)

Text-Ex 1 The position of an object moving in a plane is given by $\vec{r}(t) = t^3\,\vec{i} + t^2\,\vec{j}$. Find its velocity, speed and acceleration when $t=1$.

Ans : $\vec{v}(t) = \vec{r}'(t) = \langle 3t^2, 2t \rangle$

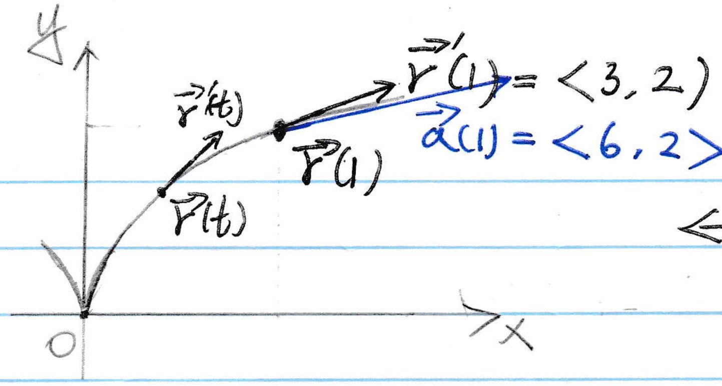
$$|\vec{v}(t)| = \sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2}$$

$$\vec{a}(t) = \vec{v}'(t) = \langle 6t, 2 \rangle$$

When $t=1$, $\vec{v}(1) = \langle 3, 2 \rangle$

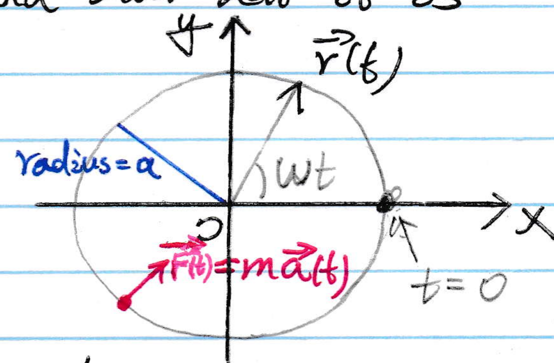
$$|\vec{v}(1)| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$\vec{a}(1) = \langle 6, 2 \rangle .$$




← Picture for Text Ex 1

<u>Text-Ex 4</u>: An object with mass $m$ (that moves in a circular path with constant angular speed $\omega$) has a position vector $\vec{r}(t) = \langle a\cos(\omega t), a\sin(\omega t) \rangle$. Find the force acting on the object and show that it is directed toward the origin.




<u>Ans</u>:

How to compute the force: $\vec{F}(t) = m\vec{a}(t)$ (Newton's second law)

To find $\vec{a}(t)$, we take derivatives

$$\vec{v}(t) = \vec{r}'(t) = \langle -a\omega \sin(\omega t), a\omega \cos(\omega t) \rangle$$

$$\vec{a}(t) = \vec{v}'(t) = \langle -a\omega^2 \cos(\omega t), -a\omega^2 \sin(\omega t) \rangle$$

Therefore

$$\vec{F}(t) = m\vec{a}(t) = \boxed{\langle -am\omega^2 \cos(\omega t), -am\omega^2 \sin(\omega t) \rangle}$$

Notice that

$$\vec{F}(t) = -m\omega^2 \vec{r}(t).$$

So the force acts in the direction opposite to the position vector $\vec{r}(t)$. Hence the force points toward the origin. (think why?)

2. Compute velocity when acceleration is known, compute position when velocity is known.

$$\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} \vec{a}(s)\, ds$$

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{v}(s)\, ds$$

Text-Ex 3 A moving particle starts at $\vec{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\vec{v}(0) = \vec{i} - \vec{j} + \vec{k} = \langle 1, -1, 1 \rangle$. Its acceleration is $\vec{a}(t) = 4t\,\vec{i} + 6t\,\vec{j} + \vec{k}$. Find its velocity and position at time $t$.

Ans: To find $\vec{r}(t)$, we need $\vec{v}(t)$ first.

Use $\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} \vec{a}(s)\, ds$ to compute $\vec{v}(t)$.

choose $t_0 = 0$ b/c $\vec{v}(0)$ is known

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \langle 4s, 6s, 1 \rangle\, ds.$$

Since $\int_0^t 4s\, ds = 2s^2 \Big|_0^t = 2t^2 - 0 = 2t^2$

$$\int_0^t 6s\, ds = 3s^2 \Big|_0^t = 3t^2 - 0 = 3t^2$$

$$\int_0^t 1\, ds = s \Big|_0^t = t - 0 = t$$

Hence

$$\vec{v}(t) = \underbrace{\langle 1, -1, 1 \rangle}_{\vec{v}(0)} + \underbrace{\langle 2t^2, 3t^2, t \rangle}_{\int_0^t \vec{a}(s)\, ds}$$

$$= \langle 2t^2 + 1,\ 3t^2 - 1,\ t + 1 \rangle.$$

Now we compute $\vec{r}(t)$ using $\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{v}(s)\,ds$.

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(s)\,ds$$

(Here we choose $t_0 = 0$ in the formula b/c $\vec{r}(0)$ is known)

$$= \vec{r}(0) + \int_0^t \langle 2s^2+1,\ 3s^2-1,\ s+1 \rangle\,ds$$

Since

$$\int_0^t (2s^2+1)\,ds = \left(\tfrac{2}{3}s^3 + s\right)\Big|_0^t = \tfrac{2}{3}t^3 + t$$

$$\int_0^t (3s^2-1)\,ds = \left(s^3 - s\right)\Big|_0^t = t^3 - t$$

$$\int_0^t (s+1)\,ds = \left(\tfrac{1}{2}s^2 + s\right)\Big|_0^t = \tfrac{1}{2}t^2 + t$$

So

$$\int_0^t \langle 2s+1,\ 3s-1,\ s+1 \rangle\,ds = \langle \tfrac{2}{3}t^3 + t,\ t^3 - t,\ \tfrac{1}{2}t^2 + t \rangle$$

and

$$\vec{r}(t) = \langle 1, 0, 0 \rangle + \langle \tfrac{2}{3}t^3 + t,\ t^3 - t,\ \tfrac{1}{2}t^2 + t \rangle$$

$$= \langle \tfrac{2}{3}t^3 + t + 1,\ t^3 - t,\ \tfrac{1}{2}t^2 + t \rangle.$$

<u>Remark:</u> An alternative approach

Indefinite integral + determine constant $\vec{C}$.

For example, when computing $\vec{v}(t)$ in the example above, we first write

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 4t, 6t, 1 \rangle\,dt$$

Then we compute $\int \langle 4t, 6t, 1\rangle\, dt$ and get

$$\vec{v}(t) = \int \langle 4t, 6t, 1\rangle\, dt = \langle 2t^2, 3t^2, t\rangle + \vec{C}$$

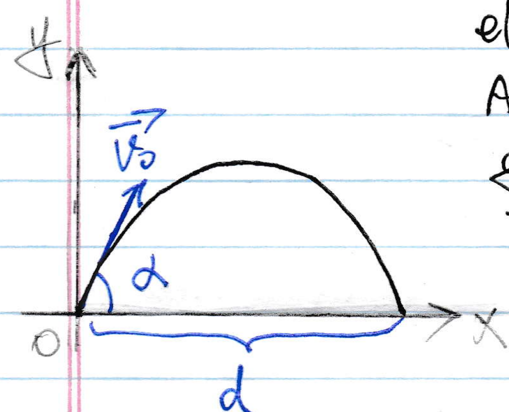
To determine $\vec{C}$, use $\vec{v}(0) = 0$ and plug $t=0$ into the equation above.

$$\langle 1, -1, 1\rangle = \vec{v}(0) = \langle 2(0)^2, 3(0)^2, 0\rangle + \vec{C}$$
$$= \vec{C}$$

Once we get $\vec{C}$, we have

$$\vec{v}(t) = \langle 2t^2, 3t^2, t\rangle + \langle 1, -1, 1\rangle$$
$$= \langle 2t^2+1, 3t^2-1, t+1\rangle.$$

<u>Text-Ex 5</u>




A projectile is fired with angle of elevation $\alpha$ and initial velocity $\vec{v_0}$. Assume the only external force is due to gravity, find the position function $\vec{r}(t)$ of the projectile. What value of $\alpha$ maximizes the range $d$?

<u>Ans</u>: Since $\vec{F} = -mg\,\vec{j}$ is the only force, we know $\vec{a}(t) = \dfrac{\vec{F}}{m} = -g\,\vec{j} = \langle 0, -g\rangle$.

So $\vec{v}(t) = \int \vec{a}(t)\, dt = \int \langle 0, -g\rangle\, dt$

$$\Rightarrow \quad \vec{v}(t) = \langle 0, -gt \rangle + \vec{C}$$

Plug $t=0$ into the equation above to determine $\vec{C}$.

$$\vec{V_0} = \vec{v}(0) = \langle 0, -g(0) \rangle + \vec{C}$$

$$\Rightarrow \quad \vec{C} = \vec{V_0}, \quad \text{hence } \vec{v}(t) = \vec{V_0} - gt\,\vec{j}$$

To compute $\vec{r}(t)$,

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \left(\vec{V_0} - gt\,\vec{j}\right) dt$$

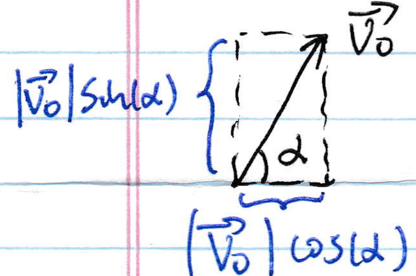
$$= \vec{V_0}\,t - \frac{1}{2} g t^2 \vec{j} + \vec{D}.$$

To determine $\vec{D}$, plug in $t=0$

$$\vec{0} = \vec{r}(0) = \vec{V_0}(0) - \frac{1}{2} g (0)^2 \vec{j} + \vec{D} = \vec{D}$$

So $\vec{D} = \vec{0}$ and $\vec{r}(t) = t\vec{V_0} - \frac{1}{2} g t^2 \vec{j}$.

Write into the component form,




$$\vec{V_0} = \langle |\vec{V_0}| \cos(\alpha), |\vec{V_0}| \sin(\alpha) \rangle$$

$$\Rightarrow \quad \vec{r}(t) = t\vec{V_0} - \frac{1}{2} g t^2 \vec{j}$$

$$= \left\langle |\vec{V_0}| \cos(\alpha)\, t,\ |\vec{V_0}| \sin(\alpha)\, t - \frac{1}{2} g t^2 \right\rangle$$

Parametric equations of the trajectory

$$x = |\vec{v_0}| \cos(\alpha)\, t, \quad y = |\vec{v_0}| \sin(\alpha)\, t - \frac{1}{2} g t^2 .$$

To find the relation between $\alpha$ and $d$, we first compute $d$. The time that the projectile hits the ground is

$$|\vec{v_0}| \sin(\alpha)\, t - \frac{1}{2} g t^2 = 0$$

Initial time when projectile is on the ground

$$t \left( |\vec{v_0}| \sin(\alpha) - \frac{1}{2} g t \right) = 0$$

$$\underline{t = 0} \quad \text{or} \quad \underline{t = \frac{2|\vec{v_0}| \sin(\alpha)}{g}}$$

So

$$d = |\vec{v_0}| \cos(\alpha) \left( \frac{2|\vec{v_0}| \sin(\alpha)}{g} \right)$$

(Plug $t = \frac{2|\vec{v_0}|\sin(\alpha)}{g}$ into $x = |\vec{v_0}| \cos(\alpha) t$)

$$= \frac{2|\vec{v_0}|^2 \cos(\alpha) \sin(\alpha)}{g}$$

$$= \frac{|\vec{v_0}|^2 \sin(2\alpha)}{g}$$

(Note $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$)

Therefore when $|\vec{v_0}|$ is fixed, $d$ has the maximum value when $\sin(2\alpha) = 1$ i.e. $\alpha = \frac{\pi}{4} = 45^\circ$.

## 3. Tangential and Normal Components of Acceleration

Split acceleration into two components, one in the direction of motion and one perpendicular to it.

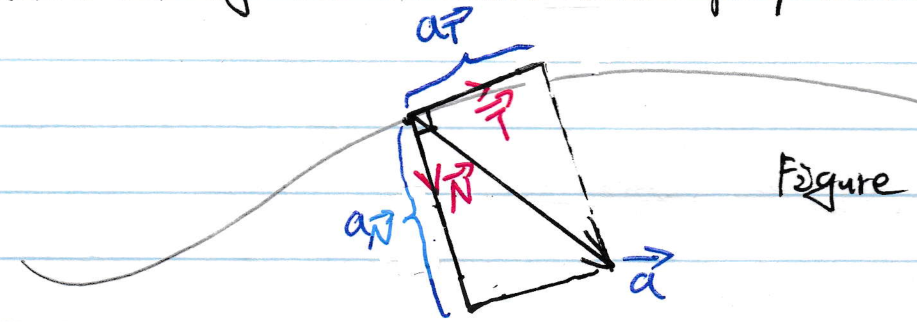



Figure 7 in the textbook

$$\vec a = a_{\vec T}\,\vec T + a_{\vec N}\,\vec N$$

where: the tangential component of acceleration is

$$a_{\vec T}(t) = \frac{\vec v(t)\cdot \vec a(t)}{|\vec v(t)|} = \frac{\vec r\,'(t)\cdot \vec r\,''(t)}{|\vec r\,'(t)|}$$

and the normal component of acceleration is

$$a_{\vec N}(t) = \kappa(t)\,|\vec v(t)|^2 = \frac{|\vec r\,'(t)\times \vec r\,''(t)|}{|\vec r\,'(t)|} = \frac{|\vec v\times\vec a|}{|\vec v|}$$

**Text-Ex 7** Find $a_{\vec T}$ and $a_{\vec N}$ for $\vec r(t) = \langle t^2, t^2, t^3\rangle$ at $t=1$ and for general $t$.

**Ans:**

$$\vec r(t) = \langle t^2,\ t^2,\ t^3\rangle$$

$$\vec v(t) = \vec r\,'(t) = \langle 2t,\ 2t,\ 3t^2\rangle$$

$$\vec a(t) = \vec v\,'(t) = \langle 2,\ 2,\ 6t\rangle$$

$$a_{\vec{T}}(t) = \frac{\vec{v}(t) \cdot \vec{a}(t)}{|\vec{v}(t)|} = \frac{(2t)(2) + (2t)(2) + (3t^2)(6t)}{\sqrt{(2t)^2 + (2t)^2 + (3t^2)^2}}$$

$$= \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$$

$$\vec{v}(t) \times \vec{a}(t) = \left\langle \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix}, - \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix}, \begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \right\rangle$$

$$= \langle 6t^2, -6t^2, 0 \rangle$$

So $$a_{\vec{N}}(t) = \frac{|\vec{v}(t) \times \vec{a}(t)|}{|\vec{v}(t)|}$$

$$= \frac{\sqrt{(6t^2)^2 + (-6t^2)^2 + 0^2}}{\sqrt{8t^2 + 9t^4}} = \frac{6\sqrt{2}\, t^2}{\sqrt{8t^2 + 9t^4}}.$$

At $t = 1$, $$a_{\vec{T}}(1) = \frac{8(1) + 18(1)^3}{\sqrt{8(1)^2 + 9(1)^4}} = \frac{26}{\sqrt{17}}$$

$$a_{\vec{N}}(1) = \frac{6\sqrt{2}\,(1)^2}{\sqrt{17}} = \frac{6\sqrt{2}}{\sqrt{17}}.$$

4. Kepler's Laws (Not required) (See the textbook if you like)

5. Summary of formulas for $\vec{r}(t)$

$$\boxed{\begin{gathered} \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}, \quad \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}, \quad \vec{B}(t) = \vec{T}(t) \times \vec{N}(t), \\ \kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}, \quad a_{\vec{N}}(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}, \\ a_{\vec{T}}(t) = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|} \end{gathered}}$$