When we discuss length, the domain should be finite in most cases, o/w the length $= \infty$.

13.3 Arc Length and Curvature

1. Length of a curve:

Suppose C is a curve with a parametrization

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle, \quad a \le t \le b$$

then the length of C is given by

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt = \int_a^b |\vec{r}\,'(t)|\, dt.$$

In 2D, for a curve C with a parametrization

$$\vec{r}(t) = \langle f(t), g(t) \rangle, \quad a \le t \le b,$$

the length of C is also given by

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt = \int_a^b |\vec{r}\,'(t)|\, dt$$

Text-Ex 1: Find the length of the arc with vector equation $\vec{r}(t) = \cos(t)\,\vec{i} + \sin(t)\,\vec{j} + t\,\vec{k}$ from the point $(1,0,0)$ to the point $(1, 0, 2\pi)$

Ans: We have the expression of $\vec{r}$, to use the arc length formula, we need a and b as well.

To find a and b, we just solve t from $\vec{r}(t) = \langle 1, 0, 0 \rangle$ and $\vec{r}(t) = \langle 1, 0, 2\pi \rangle$ respectively

$$\begin{cases} \cos(t) = 1 \\ \sin(t) = 0 \\ t = 0 \end{cases} \Rightarrow t = 0, \text{ so } a = 0$$

$$\begin{cases} \cos(t) = 1 \\ \sin(t) = 0 \\ t = 2\pi \end{cases} \Rightarrow t = 2\pi, \text{ so } b = 2\pi$$

The part of arc between $(1,0,0)$ and $(1,0,2\pi)$ is described by $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ $\quad 0 \le t \le 2\pi$.

Plug into the formula : $\mathcal{L} = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt$.

$$\vec{r}\,'(t) = \langle \underbrace{-\sin(t)}_{\frac{d}{dt}(\cos(t))}, \underbrace{\cos(t)}_{\frac{d}{dt}(\sin(t))}, \underbrace{1}_{\frac{d}{dt}(t)} \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{(-\sin(t))^2 + \cos^2(t) + 1^2} = \sqrt{1+1} = \sqrt{2}$$

So

$$\mathcal{L} = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\,(2\pi) = 2\sqrt{2}\,\pi.$$

2. The arc length function

Compute length of a curve : definite integral $\int_a^b$

Compute arc length function : definite integral $\int_a^{t}$

For the curve C given by $\vec{r}(t)$ $\quad a \le t \le b$. Arc length function

$$s(t) = \int_a^t |\vec{r}\,'(u)|\, du.$$

$s(t)$ is a function of t.

<u>Text-Ex 2</u>: Compute the arc length function $s(t)$ for the curve given by

$$\vec{r}(t) = \cos(t)\,\vec{i} + \sin(t)\,\vec{j} + t\,\vec{k} \qquad 0 \le t \le 2\pi.$$

<u>Ans</u>: We have computed already in Text-Ex 1 that

$$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle,$$

$$|\vec{r}'(t)| = \sqrt{2}.$$

So

$$s(t) = \int_a^t |\vec{r}'(u)|\,du = \int_0^t \sqrt{2}\,du$$
$$= \sqrt{2}\,t.$$

(We don't need this much, just an introduction)

3. Smooth and piecewise smooth parametrizations

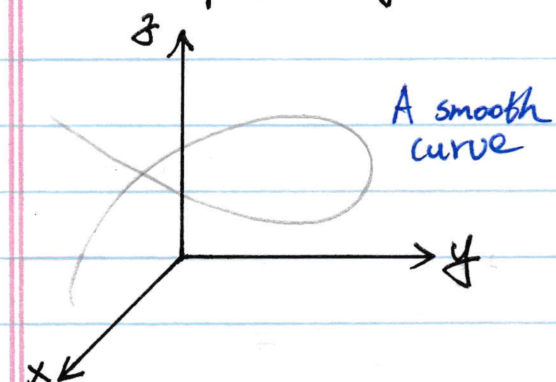
- A smooth parametrization has $\vec{r}'(t) \neq \vec{0}$ for $t \in (a,b)$

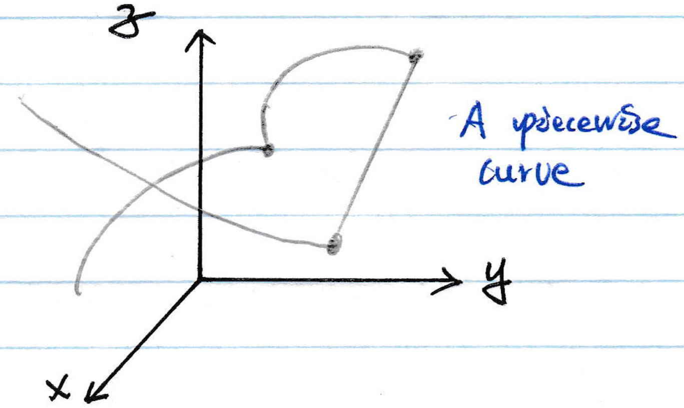
(Analogy: if $\vec{r}(t)$ is the position at time t, then velocity $= \vec{r}'(t)$, so $\vec{r}'(t) \neq \vec{0}$ means never stop)

- A piecewise smooth parametrization is a parametrization in which you can break the t-range into pieces on which the parametrization is smooth.

(Analogy: your commute)

• A curve is smooth if it has a smooth parametrization, and a curve is piecewise smooth if it has a piecewise smooth parametrization.





Example: $\vec{r}(t) = \langle t+2, 3-2t, 2t \rangle$.

$\vec{r}\,'(t) = \langle 1, -2, 2 \rangle \neq \vec{0}$. Smooth curve.

Example: $\vec{r}(t) = \langle t^2, \cos(t), 1 \rangle$ for $t \in [-1, 1]$.

$\vec{r}\,'(t) = \langle 2t, -\sin(t), 0 \rangle$.

By solving $\vec{r}\,'(t) = \vec{0}$, we find out it has a solution $t=0$. $\vec{r}\,'(0) = \vec{0}$ hence not smooth.

Example: $\vec{r}(t) = \langle t - \sin(t), 1 - \cos(t) \rangle$, $-2\pi \leq t \leq 2\pi$.

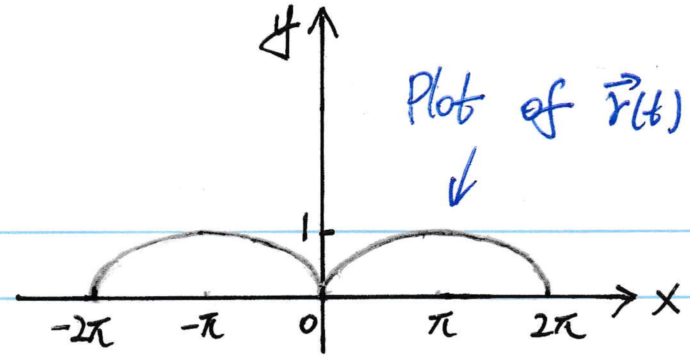
$\vec{r}\,'(t) = \langle 1 - \cos(t), \sin(t) \rangle$

$\vec{r}\,'(0) = \langle 1 - \cos(0), \sin(0) \rangle = \langle 0, 0 \rangle = \vec{0}$

Hence the curve is not smooth.

But if we cut the curve into two pieces, one for $t \in (-2\pi, 0)$, another for $t \in (0, 2\pi)$.

Since $\vec{r}\,'(t) \neq \vec{0}$ for $t \in (-2\pi, 0)$ and $t \in (0, 2\pi)$, the curve is piecewise smooth.



Plot of $\vec{r}(t) = \langle t - \sin(t), 1 - \cos(t) \rangle$
$-2\pi \le t \le 2\pi$.

4. Curvature ← tells us how much the curve deviates from being a straight line

Definition:
$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

curvature, scalar function. Recall $\vec{T}$ is the unit tangent vector

Text-Ex 3 Show that the curvature of a circle of radius a is $1/a$. (large circle small curvature)

Ans: For simplicity, consider a circle with its center at the origin. We take a parametrization of it:

$$\vec{r}(t) = \langle a\cos(t), a\sin(t) \rangle.$$

Recall $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$, so we need to compute

$$\vec{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle$$

$$|\vec{r}'(t)| = \sqrt{(-a\sin(t))^2 + (a\cos(t))^2} = \sqrt{a^2} = a$$

$$\vec{T}(t) = \frac{\langle -a\sin(t), a\cos(t) \rangle}{a} = \langle -\sin(t), \cos(t) \rangle$$

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{1}{a}|\vec{T}'(t)|.$$ To find $\vec{T}'(t)$,

$$\vec{T}'(t) = \langle -\cos(t), -\sin(t) \rangle, \quad |\vec{T}'(t)| = \sqrt{(-\cos(t))^2 + (-\sin(t))^2} = 1$$

$$\Rightarrow \kappa(t) = \frac{1}{a}(1) = \boxed{\frac{1}{a}}.$$

Sometimes it's easier to use the formula

$$k(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

to compute the curvature.

Text - Ex 4 Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at a general point and at $(0,0,0)$.

Ans: In the formula above for $k(t)$, we need $\vec{r}\,'(t), \vec{r}\,''(t)$.

$$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle, \quad \vec{r}\,''(t) = \langle 0, 2, 6t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \left\langle \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix}, -\begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix}, \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} \right\rangle$$

$$\begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} = (2t)(6t) - 3t^2(2) = 12t^2 - 6t^2 = 6t^2$$

$$\begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} = 1(6t) - 3t^2(0) = 6t - 0 = 6t$$

$$\begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} = 1(2) - 2t(0) = 2 - 0 = 2$$

So $\vec{r}\,'(t) \times \vec{r}\,''(t) = \langle 6t^2, -6t, 2 \rangle$

$$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{(6t^2)^2 + (-6t)^2 + 2^2}$$

$$= \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$$

Hence $$k(t) = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

To get the curvature at $(0,0,0)$. Notice that $\vec{r}(0) = \langle 0,0,0 \rangle$ hence at $(0,0,0)$ $t=0$.

Plug $t=0$ into the formula of $k(t)$:

$$k(0) = \frac{2\sqrt{9(0)^4+9(0)^2+1}}{(1+4(0)^2+9(0)^4)^{3/2}} = \frac{2}{1} = 2.$$

5. Curvature formula for the graph of $y=f(x)$ in 2D.

$$\underset{\uparrow}{k(x)} = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}.$$

curvature of the curve at point $(x, f(x))$.

This formula can be derived using the parametrization $\vec{r}(x) = \langle x, f(x) \rangle$ and the formula

$$k(x) = \frac{|\vec{r}'(x) \times \vec{r}''(x)|}{|\vec{r}'(x)|^3}$$

Text-Ex 5 Find the curvature of $y=x^2$ at points $(0,0)$, $(1,1)$ and $(2,4)$.

Ans: We use our formula to compute $k(x)$ first.

$f(x)=x^2$, $f'(x) = 2x$, $f''(x) = 2$

$$\Rightarrow \quad k(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \frac{|2|}{(1+(2x)^2)^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$$

To find the curvature at given points, notice that $(0,0)$, $(1,1)$, $(2,4)$ correspond to $x=0, 1, 2$.

$$k(0) = \frac{2}{(1+4(0)^2)^{3/2}} = \frac{2}{1} = 2$$

$$k(1) = \frac{2}{(1+4(1)^2)^{3/2}} = \frac{2}{5^{3/2}}$$

$$k(2) = \frac{2}{(1+4(2)^2)^{3/2}} = \frac{2}{17^{3/2}}$$

6. Normal and Binormal Vectors, Normal Plane

Normal vector $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$

Binormal Vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.

Both normal vector and binormal vector are perpendicular to the (unit) tangent vector. $\vec{N}(t) \perp \vec{T}(t)$, $\vec{B}(t) \perp \vec{T}(t)$

Normal plane at point P (corresponding to $\vec{r}(t_0)$)

$$\left(\langle x, y, z\rangle - \vec{r}(t_0)\right) \cdot \vec{r}\,'(t_0) = 0$$

This equation gives the plane containing P and $\perp$ $\vec{r}\,'(t_0)$.

Text-Ex 6: Find the unit normal and binormal vectors for the circular helix $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

Ans: We need $\vec{T}(t)$ in order to compute $\vec{N}(t)$, $\vec{B}(t)$.

$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$, $|\vec{r}'(t)| = \sqrt{(-\sin(t))^2 + \cos^2(t) + 1^2} = \sqrt{2}$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$$

$$\vec{T}'(t) = \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle$$

$$|\vec{T}'(t)| = \frac{1}{\sqrt{2}} \sqrt{(-\cos(t))^2 + (-\sin(t))^2 + 0^2} = \frac{1}{\sqrt{2}}$$

So $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle -\cos(t), -\sin(t), 0 \rangle$.

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{2}} \left\langle \begin{vmatrix} \cos(t) & 1 \\ -\sin(t) & 0 \end{vmatrix}, -\begin{vmatrix} -\sin(t) & 1 \\ -\cos(t) & 0 \end{vmatrix}, \begin{vmatrix} -\sin(t) & \cos(t) \\ -\cos(t) & -\sin(t) \end{vmatrix} \right\rangle$$

$$= \frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), 1 \rangle.$$

Text-Ex 7 Find the equation of the normal plane for the curve in Text-Ex 6 at the point $P(0, 1, \frac{\pi}{2})$

Ans: Point P corresponds to $t = \frac{\pi}{2}$. $(\vec{r}(\frac{\pi}{2}) = \langle 0, 1, \frac{\pi}{2} \rangle)$

The normal plane is orthogonal to $\vec{r}'(\frac{\pi}{2}) = \langle -\sin(\frac{\pi}{2}), \cos(\frac{\pi}{2}), 1 \rangle$

$= \langle -1, 0, 1 \rangle$

So the equation is $(-1)(x-0) + (0)(y-1) + (1)(z - \frac{\pi}{2}) = 0$

$$\boxed{-x + z - \frac{\pi}{2} = 0}$$

7. Topics not required: parametrize a curve with respect to arc length, osculating circle.