13.2 Derivatives and Integrals of Vector Functions

1. Definition: the derivative $\vec{r}\,'$ of a vector function $\vec{r}$ is found by taking the derivative of the component functions. For $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$,

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$
$$= f'(t)\,\vec{i} + g'(t)\,\vec{j} + h'(t)\,\vec{k}$$

Text-Ex 1: (a) Find the derivative of

$$\vec{r}(t) = (1+t^3)\,\vec{i} + te^{-t}\,\vec{j} + \sin(2t)\,\vec{k}$$

Ans: $\vec{r}\,'(t) = \langle (1+t^3)', (te^{-t})', (\sin(2t))' \rangle$

$(1+t^3)' = 3t^2$,

$(te^{-t})' = (1)e^{-t} + t(-e^{-t}) = e^{-t}(1-t)$

$(\sin(2t))' = \cos(2t)\cdot(2) = 2\cos(2t)$

So $\vec{r}\,'(t) = \langle 3t^2, (1-t)e^{-t}, 2\cos(2t) \rangle$

2. Properties of Derivatives: (c is scalar, f is a real-valued function, $\vec{u}$ and $\vec{v}$ are vector functions)

(1) $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}\,'(t) + \vec{v}\,'(t)$

(2) $\frac{d}{dt}[c\,\vec{u}(t)] = c\,\vec{u}\,'(t)$

(3) $\frac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$

(4) $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

(5) $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

(6) $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\vec{u}'(f(t))$

Chain rule

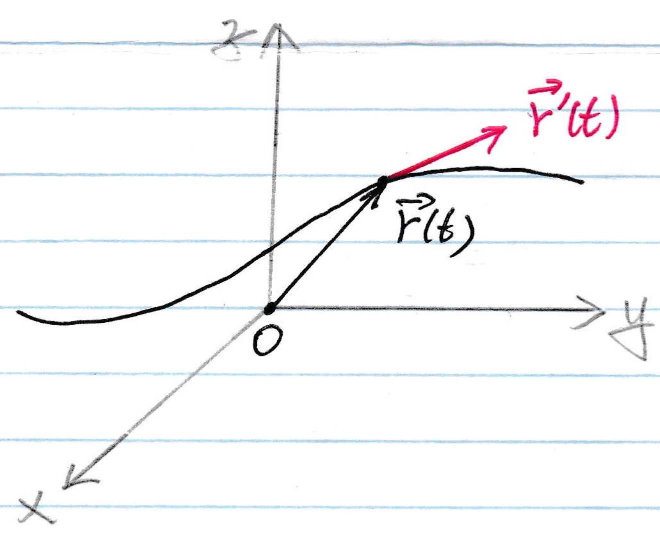
Good to know these properties, but we don't use them much.

3. Applications of $\vec{r}'(t)$: Tangent Vector

The tangent line to the curve at $\vec{r}(t)$ is parallel to the vector $\vec{r}'(t)$. So $\vec{r}'(t)$ is called the tangent vector.

We sometimes also consider the unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$



<u>Text-Ex 1 (b)</u> : Find the unit tangent vector at the point where $t=0$ for the curve

$$\vec{r}(t) = \langle 1+t^3 ,\ te^{-t} ,\ \sin(2t) \rangle$$

<u>Ans</u>: We have computed in the example before that

$$\vec{r}'(t) = \langle 3t^2 ,\ (1-t)e^{-t} ,\ 2\cos(2t) \rangle$$

So $\vec{r}'(0) = \langle 3(0)^2 ,\ (1-0)e^{-0} ,\ 2\cos(2(0)) \rangle$

$$= \langle 0 , 1 , 2 \rangle$$

Therefore $\vec{T}(0) = \dfrac{\vec{r}'(0)}{|\vec{r}'(0)|} = \dfrac{\langle 0, 1, 2 \rangle}{\sqrt{0^2+1^2+2^2}}$

$$= \frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle = \langle 0 , \frac{1}{\sqrt{5}} , \frac{2}{\sqrt{5}} \rangle .$$

4. Indefinite Integrals

The indefinite integral of $\vec{r}(t)$ is computed by calculating the indefinite integrals of its component functions. For $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt ,\ \int g(t)\,dt ,\ \int h(t)\,dt \right\rangle$$

$$= \left(\int f(t)\,dt\right)\vec{i} + \left(\int g(t)\,dt\right)\vec{j} + \left(\int h(t)\,dt\right)\vec{k}$$

We have "$+\vec{C}$" at the end rather than giving each components its own constant.

<u>Example</u>: Compute $\int \vec{r}(t)\,dt$ for

$$\vec{r}(t) = 2\cos(t)\,\vec{i} + \sin(t)\,\vec{j} + 2t\,\vec{k}.$$

<u>Ans</u>: $\int \vec{r}(t)\,dt = \left(\int 2\cos(t)\,dt\right)\vec{i} + \left(\int \sin(t)\,dt\right)\vec{j} + \left(\int 2t\,dt\right)\vec{k}$

$$= \left(2\sin(t) + C_1\right)\vec{i} + \left(-\cos(t) + C_2\right)\vec{j} + \left(t^2 + C_3\right)\vec{k}$$

$$= 2\sin(t)\,\vec{i} - \cos(t)\,\vec{j} + t^2\vec{k} + \underbrace{\left(C_1\vec{i} + C_2\vec{j} + C_3\vec{k}\right)}_{\vec{C}}$$

$$= 2\sin(t)\,\vec{i} - \cos(t)\,\vec{j} + t^2\vec{k} + \vec{C}$$

or $= \langle 2\sin(t), -\cos(t), t^2 \rangle + \vec{C}$.

5. Definite Integrals

We take definite integrals for each compoent function respectively.

For $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$,

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}.$$

We can also extend the Fundamental Theorem of Calculus to vector function as follows

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

where $\vec{R}$ is an indefinite integral of $\vec{r}(t)$. However, we don't use this much. We do integrals for each component functions and use Fundamental Theorem of Calculus for scalar functions.

<u>Text-Ex 5</u>: Compute $\int_0^{\frac{\pi}{2}} \vec{r}(t)\,dt$ for

$$\vec{r}(t) = 2\cos(t)\,\vec{i} + \sin(t)\,\vec{j} + 2t\,\vec{k}$$

<u>Ans</u> We have already computed an antiderivative of $\vec{r}$ in the previous example:

$$\vec{R}(t) = \langle 2\sin(t),\ -\cos(t),\ t^2 \rangle.$$

So one way is to use the Fundamental Theorem of Calculus:

$$\int_0^{\frac{\pi}{2}} \vec{r}(t)\,dt = R\left(\tfrac{\pi}{2}\right) - R(0)$$
$$= \langle 2\sin(\tfrac{\pi}{2}), -\cos(\tfrac{\pi}{2}), (\tfrac{\pi}{2})^2 \rangle - \langle 2\sin(0), -\cos(0), 0^2 \rangle$$
$$= \langle 2, 0, \tfrac{\pi^2}{4} \rangle - \langle 0, -1, 0 \rangle$$
$$= \langle 2, 1, \tfrac{\pi^2}{4} \rangle = 2\vec{i} + \vec{j} + \tfrac{\pi^2}{4}\vec{k}.$$

If we don't know the antiderivative of $\vec{r}(t)$, another way to do the calculation is integrating each component function independently:

$$\int_0^{\frac{\pi}{2}} 2\cos(t)\,dt = 2\sin(t)\Big|_0^{\frac{\pi}{2}} = 2-0=2$$

$$\int_0^{\frac{\pi}{2}} \sin(t)\,dt = \left(-\cos(t)\right)\Big|_0^{\frac{\pi}{2}} = 0-(-1) = 1$$

$$\int_0^{\frac{\pi}{2}} 2t\,dt = t^2\Big|_0^{\frac{\pi}{2}} = \left(\tfrac{\pi}{2}\right)^2 - 0^2 = \frac{\pi^2}{4}$$

Therefore

$$\int_0^{\frac{\pi}{2}} \vec{r}(t)\,dt = \langle 2, 1, \tfrac{\pi^2}{4} \rangle.$$