Math 241 Section 13.1 Vector Functions and Space Curves

1. Definition of vector-valued function.
A **vector-valued function**, or **vector function**, is a function where a number (a parameter, typically t) goes in and a vector comes out. Typical notation in 3D:

$$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k} = \langle f(t), g(t), h(t)\rangle$$

often with a range of t given. f, g and h are real-valued functions called the **component functions** of $\boldsymbol{r}$. In 2D, we do not have the third component, and the notation looks like

$$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} = \langle f(t), g(t)\rangle.$$

We can also think $\boldsymbol{r}(t)$ as a point in 3D or 2D, with components the same as the vector $\boldsymbol{r}(t)$ (anchor the other side of the vector at the origin).

Example. Consider the following vector function in $2D$

$$\boldsymbol{r}(t) = \langle 1 - t, 2t + 3\rangle.$$

This is a line! It can also be represented by the equation $y = 5 - 2x$.

Text-Example 1. Consider the vector function

$$\boldsymbol{r}(t) = \langle t^3, \ln(3 - t), \sqrt{t}\rangle.$$

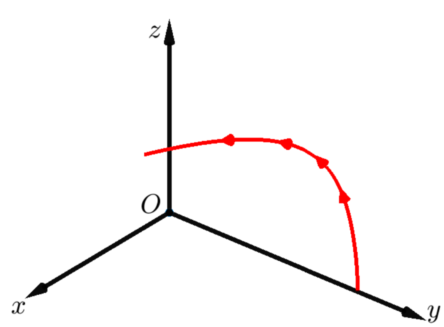
Although the domain of $\boldsymbol{r}$ is not given explicitly, by convention it consists of all values of t such that $t^3, \ln(3 - t), \sqrt{t}$ are all defined. Therefore the domain of $\boldsymbol{r}$ is $[0, 3)$.

2. Graph of vector-valued functions, more examples.
To graph $\boldsymbol{r}$, we anchor the vector at the origin and plot the endpoint (treat $\boldsymbol{r}$ as a point). We don't ask students to draw some nontrivial vector functions below, but having an idea of what the graphs look like will be helpful.

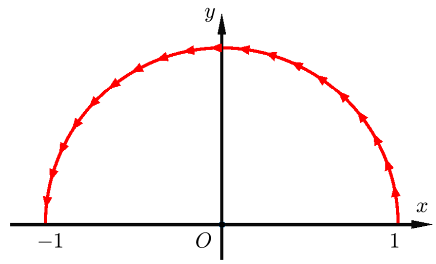
Example. Draw the vector function in Text-Example 1:

$$\boldsymbol{r}(t) = \langle t^3, \ln(3 - t), \sqrt{t}\rangle.$$

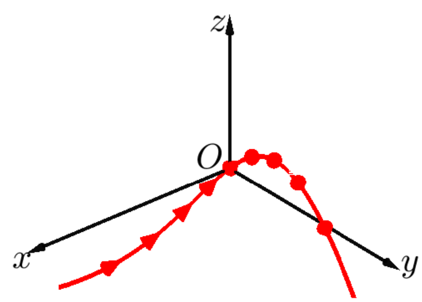
for $0 \le t \le 1$. See the graph below. The arrows on the curve indicate how the point $\boldsymbol{r}(t)$ moves as t increases.
When $t = 0$, we have $\boldsymbol{r}(0) = \langle 0, \ln(3), 0\rangle$; and when $t = 1$, we have $\boldsymbol{r}(1) = \langle 1, \ln(2), 1\rangle$. So the endpoints of the curve are $(0, \ln(3), 0)$ and $(1, \ln(2), 1)$.



Example. $r(t) = \cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j}$ with $0 \le t \le \pi$ in 2D. Its graph is a semicircle in 2D.



Example. $r(t) = t^2\boldsymbol{i} + e^t\sin(t)\boldsymbol{j} + t\cos(t)\boldsymbol{k}$ with $-1 \le t \le 1$. See the graph below, not a curve we are familiar with.



3. Limits and continuity.
The **limit** of a vector function $\boldsymbol{r}$ is fond by taking the limit of the component functions. A vector function $\boldsymbol{r}$ is **continuous at** $t = a$ if all the component functions are continuous at $t = a$. These won't be used much but still good to keep in mind.

Text-Ex 2. Find $\lim_{t\to 0} \boldsymbol{r}(t)$ where

$$\boldsymbol{r}(t) = (1 + t^3)\boldsymbol{i} + te^{-t}\boldsymbol{j} + \frac{\sin(t)}{t}\boldsymbol{k}$$

Solution:

$$\lim_{t\to 0} \boldsymbol{r}(t) = \left(\lim_{t\to 0}(1 + t^3)\right)\boldsymbol{i} + \left(\lim_{t\to 0} te^{-t}\right)\boldsymbol{i} + \left(\lim_{t\to 0}\frac{\sin(t)}{t}\right)\boldsymbol{k} = (1)\boldsymbol{i} + (0)\boldsymbol{j} + (1)\boldsymbol{k} = \boldsymbol{i} + \boldsymbol{k}.$$

4. Space curves.
Space curves are closely related to continuous vector functions. We have seen that the graph of vector function $\boldsymbol{r}$ is usually a curve.
More specifically, for $\boldsymbol{r}(t) = \langle f(t), g(t), h(t)\rangle$, the set C of all points (x, y, z) where

$$x = f(t), \quad y = g(t), \quad z = h(t) \tag{1}$$

and t varies throughout an interval I, is called a **space curve**. And the equation (1) is called **parametric equations of curve** C and t is called a **parameter**.

5. Parameterization of the curve is not unique.
Consider the curve in 2D with the following parameterization

$$\boldsymbol{r}(t) = \langle 1 - t^3, 2t^3 + 3 \rangle$$

This is a line $y = 5 - 2x$, and we can also parameterize it as

$$\boldsymbol{r}(t) = \langle 1 - t, 2t + 3 \rangle.$$

6. Finding the parametric equations of a curve.
A useful proposition: The line segment from $\boldsymbol{r}_0$ to $\boldsymbol{r}_1$ is given by (note that the textbook states this proposition in section 12.5)

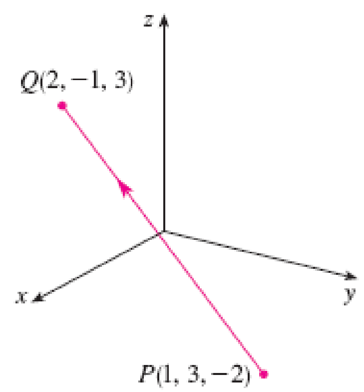
$$\boldsymbol{r}(t) = (1 - t)\boldsymbol{r}_0 + t\boldsymbol{r}_1, \quad 0 \le t \le 1.$$

Text-Ex 5. Find a vector equation and parametric equations for the line segment that joins the point $P(1, 3, -2)$ to the point $Q(2, -1, 3)$.
Solution: Using the formula above, the vector equation is

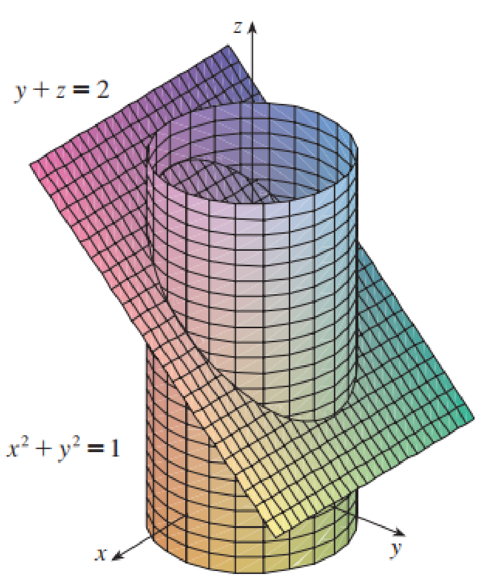
$$\begin{aligned} \boldsymbol{r}(t) &= (1 - t)\boldsymbol{r}_0 + t\boldsymbol{r}_1 = (1 - t)\langle 1, 3, -2 \rangle + t\langle 2, -1, 3 \rangle \\ &= \langle 1 + t,\ 3 - 4t,\ 5t - 2 \rangle \end{aligned}$$

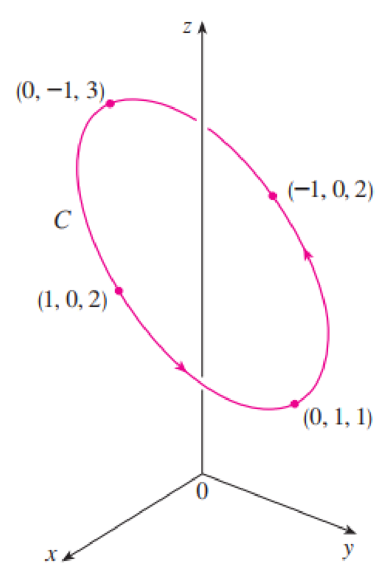
with $0 \le t \le 1$. The corresponding parametric equations are

$$x = 1 + t, \quad y = 3 - 4t, \quad z = 5t - 2 \qquad 0 \le t \le 1.$$



Text-Ex 6. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.





Solution: Since the equation of the cylinder does not involve z, we can proceed the problem as below. First, write down the parametrization of $x^2 + y^2 = 1$:

$$x = \cos(t), \quad y = \sin(t) \qquad 0 \leq t \leq 2\pi.$$

Now represent z as a function of t using the other equation $y + z = 2$:

$$z = 2 - y = 2 - \sin(t).$$

Combine the above together to get the parametrization equations

$$x = \cos(t), \quad y = \sin(t), \quad z = 2 - \sin(t) \qquad 0 \leq t \leq 2\pi.$$

The corresponding vector equation is:

$$\boldsymbol{r}(t) = \langle \cos(t),\ \sin(t),\ 2 - \sin(t) \rangle \qquad 0 \leq t \leq 2\pi.$$