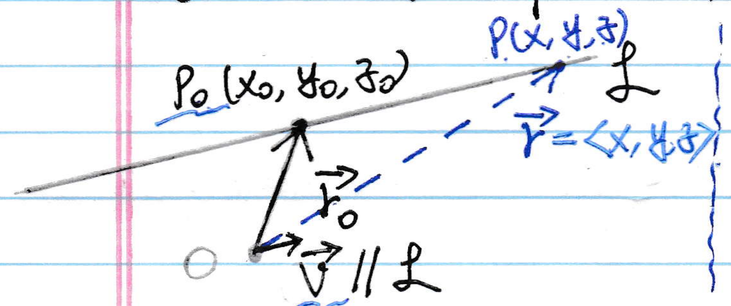
12.5 Equations of Lines and Planes

Overview : We'll discuss equations of lines first and then equations of planes. There are three different ways to construct equations of lines, all of which have their own use.

1. Vector Equations for a Line

"A point + a direction" determine a line.

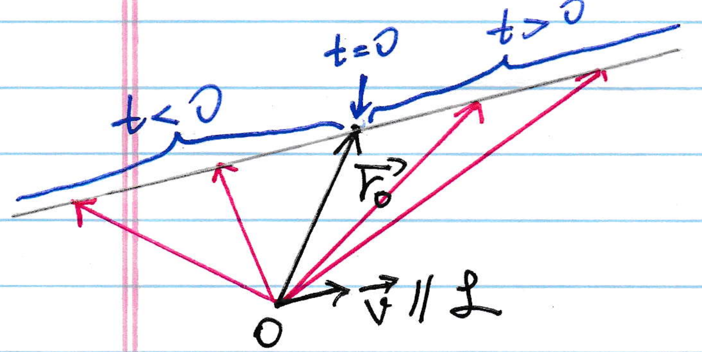
If line L is parallel to $\vec{v} = \langle a, b, c \rangle$ and



$P_0(x_0, y_0, z_0)$ is a point on line L.
Then for any point $P(x, y, z)$ on line L, we have

$$\boxed{\vec{r} = \vec{r}_0 + t\vec{v}} \quad \text{(Vector form)}$$

where t is a parameter. Each value of t (t is a real number) gives the position vector $\vec{r}$ of a point on line L.



$\vec{r} = \vec{r}(t)$ is a function of t, so for a given t, we get one point.

In the vector form, we don't see x, y, z directly.
However, $\vec{r} = \langle x, y, z \rangle$, so it does contain x, y, z in the equation.

2. Parametric Equations for a Line

In the vector form, since

$\vec{r} = \langle x, y, z \rangle$, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$, $\vec{v} = \langle a, b, c \rangle$,

so it is equivalent to

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$
$$= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

$$\Rightarrow \boxed{\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}} \quad \text{(Parametric form)}$$

Each choice of parameter t gives us a point (x, y, z) on L.

Text-Ex 1: (a) Find a vector equation and parametric equations for the line that passes through point $(5, 1, 3)$ and is parallel to the vector $\vec{i} + 4\vec{j} - 2\vec{k}$

(b) Find two other points on the line

Ans: (a) Vector eq. $\vec{r_0} = \langle 5, 1, 3 \rangle$, $\vec{v} = \langle 1, 4, -2 \rangle$

So $$\vec{r} = \vec{r_0} + t\vec{v} = \langle 5, 1, 3 \rangle + t\langle 1, 4, -2 \rangle$$

$= \langle 5+t,\ 1+4t,\ 3-2t \rangle$

So the vector equation is

$$\vec{r}(t) = \langle 5+t,\ 1+4t,\ 3-2t \rangle$$

This can be written as (look at each component

$$\begin{cases} x = 5+t \\ y = 1+4t \\ z = 3-2t \end{cases},$$

which are parametric equations of the line.

(b) To find other points, we only need to choose different t. Let

$t=1$: $x = 5+1 = 6$, $y = 1+4(1) = 5$, $z = 3-2(1) = 1$

point $(6, 5, 1)$ on the line

$t=-1$: $x = -4$, $y = -3$, $z = 5$

point $(4, -3, 5)$ on the line

(Recall that if we choose $t=0$ we will get point $(5, 1, 3)$ already given by the problem)

3. Symmetric Equations for a Line

From the parametric form $\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$, if

$a \neq 0$, $b \neq 0$, $c \neq 0$, then we can solve t in each equation

$$x = x_0 + ta \implies t = \frac{x - x_0}{a}$$

$$y = y_0 + tb \implies t = \frac{y - y_0}{b}$$

$$z = z_0 + tc \implies t = \frac{z - z_0}{c}$$

This leads to the symmetric form

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}} \; (*) \qquad \left(\text{symmetric form}\right)$$

<u>Example</u>: Rewrite the equation of L in Text-Ex 1 into the symmetric form.

<u>Ans</u>: Parametric form $\begin{cases} x = 5+t \\ y = 1+4t \\ z = 3-2t \end{cases}$

So $t = \boxed{\frac{x-5}{1} = \frac{y-1}{4} = \frac{z-3}{-2}}$ ← symmetric form

Another way is using the formula $(*)$ and plug $(x_0, y_0, z_0) = (5, 1, 3)$ and $\langle a, b, c\rangle = \langle 1, 4, -2\rangle$ into it: $\frac{x-5}{1} = \frac{y-1}{4} = \frac{z-3}{-2}$.

It may happen that one or two of a, b, c are 0.

If $a = 0$, from the parametric form

$$\begin{cases} x = x_0 + ta = x_0 \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

Symmetric form if $a=0$, $b \neq 0$, $c \neq 0$.

we get: $\boxed{x = x_0, \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}}$

If $a=b=0$, then no need to solve t, we get

$\boxed{x = x_0, \quad y = y_0}$ ← symmetric form if $a=b=0$.

Notice that z is not mentioned in the equation above, this means z can be anything.

Example: Find parametric equations and symmetric equations for the line

(a) (Text-Exercise 9) The line through the points $A(-8,1,4)$ and $B(3,-2,4)$

(b) The line through the point $A(-8,1,4)$ and parallel to $\vec{j} = \langle 0,1,0\rangle$.

Ans: (a) We discussed how to get the equations of a line from one point and one direction.

But here we have two points but no directions.

How to solve? Use $\overrightarrow{AB}$ as the direction.

Choose $\vec{v} = \overrightarrow{AB} = \langle 11, -3, 0\rangle$

$$(x_0, y_0, z_0) = (-8, 1, 4),$$

then we get parametric equations

$$\begin{cases} x = -8 + 11t \\ y = 1 + (-3)t \\ z = 4 \end{cases}$$

From the equations above, solve t:

$$t = \frac{x+8}{11} = \frac{y-1}{-3},$$

and obtain the symmetric equations

$$z = 4, \quad \frac{x+8}{11} = \frac{y-1}{-3}.$$

(b) For Parametric Equations, $(x_0, y_0, z_0) = (-8, 1, 4)$ and $\vec{v} = \vec{j} = \langle 0, 1, 0 \rangle$, so we get

$$\begin{cases} x = -8 \\ y = 1 + t \\ z = 4 \end{cases}$$

In this case, no need to solve t, the symmetric equations are $\boxed{x = -8, \quad z = 4}$.

4. Equations for a Line are not Unique

If we change the given point to another point on the same line L, or change $\vec{v}$ to another vector parallel to $\vec{v}$ (e.g. $2\vec{v}$), we still have the same line. However, all three forms of the equations for L would be different.

In Text-Ex 1, the following equations are the parametric equations of L as well.

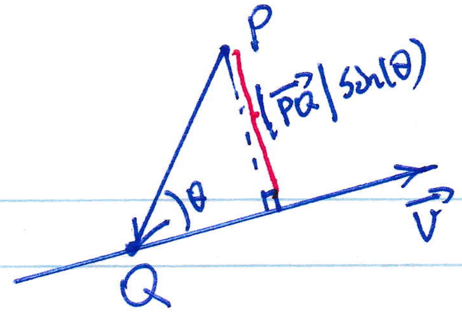
$$\begin{cases} x = 6 + 2t \\ y = 5 + 8t \\ z = 1 - 4t \end{cases}$$

(Point $(6, 5, 1)$
$\vec{v} = \langle 2, 8, -4 \rangle$)

5. Distance Formula from Point to Line:

Consider a line with direction vector $\vec{v}$ and assume point Q is on the line. Then the distance from another point P to the line is

$$\text{distance} = \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|}$$



Text-Exercise 69: Find the distance from $P(4,1,-2)$ to the line $\mathcal{L}$ with parametric equations:

$x = 1+t, \quad y = 3-2t, \quad z = 4-3t.$

Ans: From the equations for $\mathcal{L}$, we see $\vec{v} = \langle 1, -2, -3 \rangle$ is a direction vector of $\mathcal{L}$ and $Q(1,3,4)$ is on $\mathcal{L}$ (Why?).

To use the distance formula, we compute

$$\vec{PQ} = \langle -3, 2, 6 \rangle,$$

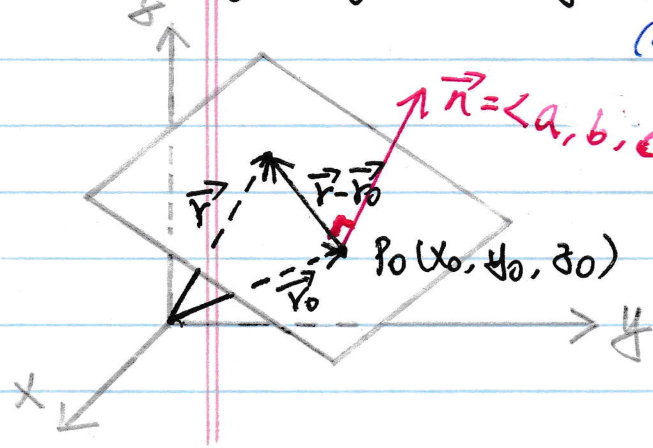
$$\vec{PQ} \times \vec{v} = \langle 6, -3, 4 \rangle$$

and thus

$$\text{distance} = \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|} = \frac{|\langle 6,-3,4 \rangle|}{|\langle 1,-2,-3 \rangle|}$$

$$= \frac{\sqrt{6^2+(-3)^2+4^2}}{\sqrt{1^2+(-2)^2+(-3)^2}} = \frac{\sqrt{61}}{\sqrt{14}}$$

6. Equations of Planes



"A point + a direction perpendicular to the plane" determine a plane

From the picture we have

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \quad \text{or equivalently} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

Either equation is called a vector equation of the plane

Plug $\vec{n} = \langle a, b, c \rangle$ and $\vec{r} - \vec{r_0} = \langle x - x_0, y - y_0, z - z_0 \rangle$ into the equation above :

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

or equivalently $\boxed{ax + by + cz + d = 0}$

where $d = -(ax_0 + by_0 + cz_0)$. Either equation is called a scalar equation of the plane.

Text-Ex 4 : Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\vec{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

Ans Plugging $\vec{n} = \langle a, b, c \rangle = \langle 2, 3, 4 \rangle$

and $(x_0, y_0, z_0) = (2, 4, -1)$

into the scalar equation, we have

$$2(x-2) + 3(y-4) + 4(z-(-1)) = 0$$

or

$$2x + 3y + 4z = 12.$$

To find the x-intercept (equivalently we are finding the intersection of the plane with the x-axis), we set $y = z = 0$ in the equation of the plane :

$2x + 3(0) + 4(0) = 12 \Rightarrow 2x = 12 \Rightarrow x = 6.$

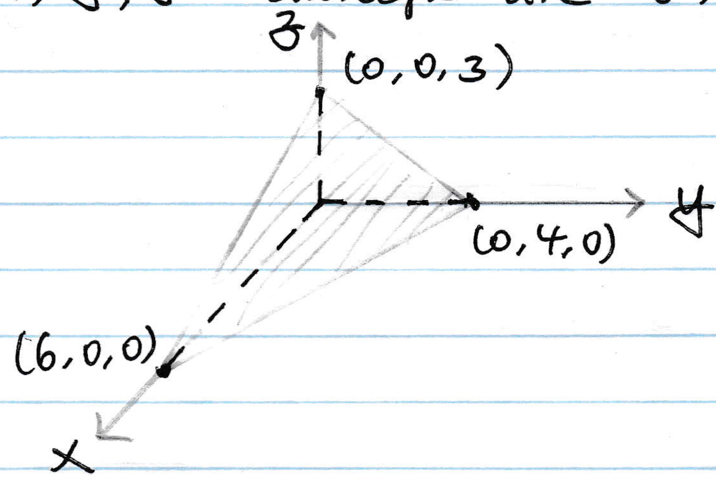
Similarly for y-intercept, we set $x = z = 0$:

$$2(0) + 3y + 4(0) = 12 \Rightarrow y = 4$$

For z-intercept. setting $x = y = 0$ gives

$2(0) + 3(0) + 4z = 12 \Rightarrow z = 3$

So x, y, z - intercepts are $6, 4, 3$ respectively



Text-Ex 5: Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$.

Ans: We need to have a vector $\vec{n}$ perpendicular to this plane. This can be done by choosing

$$\vec{n} = \vec{PQ} \times \vec{PR} \quad (\text{Why?})$$

To compute, $\vec{PQ} = \langle 2,-4,4 \rangle$. $\vec{PR} = \langle 4,-1,-2 \rangle$,

then

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$\begin{vmatrix} -4 & 4 \\ -1 & -2 \end{vmatrix} = (-4)(-2) - 4(-1) = 12$$

$$\begin{vmatrix} 2 & 4 \\ 4 & -2 \end{vmatrix} = 2(-2) - 4(4) = -20$$

$$\begin{vmatrix} 2 & -4 \\ 4 & -1 \end{vmatrix} = 2(-1) - (-4)(4) = 14$$

So $\vec{n} = \langle 12, -(-20), 14 \rangle = \langle 12, 20, 14 \rangle$.

Plugging into the equation and choosing $(x_0, y_0, z_0) = \underbrace{(1, 3, 2)}_{\text{point } P}$, we get

$$12(x-1) + 20(y-3) + 14(z-2) = 0.$$

$$\boxed{6x + 10y + 7z = 50} \leftarrow \text{equation of the plane}$$

7. Distance formula from point to plane:

Consider the plane has a point P_0 and normal vector $\vec{n}$, and another point P_1, then the distance from P_1 to the plane is

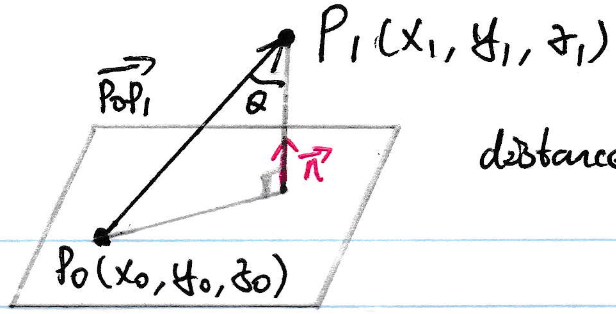
$$\text{distance} = \frac{|\vec{n} \cdot \vec{P_0P_1}|}{|\vec{n}|}.$$

If $P_0(x_0, y_0, z_0)$, $P_1(x_1, y_1, z_1)$, the plane has equation:

$$ax + by + cz + d = 0,$$

then

$$\boxed{\text{distance} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$



$$\text{distance} = |\vec{P_0P_1}| \cos(\theta)$$

Text-Ex 9: Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$

Ans: distance between planes = distance from one point (on one plane) to another plane

To find a point on the plane $10x + 2y - 2z = 5$, simply choosing $y = z = 0$, we solve x and get $x = \frac{1}{2}$.

So point $(\frac{1}{2}, 0, 0)$ is on the plane

Plugging into the distance formula:

$$\text{dist} = \frac{|5(\frac{1}{2}) + 1(0) + (-1)(0) - 1|}{\sqrt{5^2 + 1^2 + (-1)^2}}$$

We have $ax + by + cz + d = 0$ be $5x + y - z - 1 = 0$ in the formula.

$$= \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

8. Other Topics (briefly discussed in the recitation)

(1) The line segment from $\vec{r_0}$ to $\vec{r_1}$ is given by the vector equation: $\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}$, $0 \le t \le 1$

(2) How to sketch planes.