12.4 The Cross Product

1. Definition of the Determinant

A Determinant of order 2 : $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

A Determinant of order 3 :

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

↳ A determinant of order 2.

Examples: $\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 8-(-6) = 14$

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1\big(0(2) - 1(4)\big) - 2\big(3(2) - 1(-5)\big) + (-1)\big(3(4) - 0(-5)\big)$$

$$= 1(-4) - 2(11) + (-1)(12) = -38$$

2. Definition of the Cross Product

For $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$,

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$= \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle$$

When doing the calculation, maybe it's better to deal with each component separately instead of the whole formula.

Text-Ex 1 : For $\vec{a} = \langle 1, 3, 4 \rangle$, $\vec{b} = \langle 2, 7, -5 \rangle$, compute $\vec{a} \times \vec{b}$.

Ans : $\begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} = 3(-5) - 4(7) = -43$

$\begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} = 1(-5) - 4(2) = -13$

$\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} = 1(7) - 3(2) = 1$

So $\vec{a} \times \vec{b} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle$

$= \langle -43, -(-13), 1 \rangle = \langle -43, 13, 1 \rangle$

Another way to memorize the formula of $\vec{a} \times \vec{b}$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Cautions : (1) The cross product is defined only when both $\vec{a}$ and $\vec{b}$ are 3-dimensional vectors! Not valid to consider : $1 \times \langle 1, 2, 3 \rangle$ ✗ or $\langle 1, 2 \rangle \times \langle 2, 3 \rangle$ ✗

(2) Negative sign in the definition

(3) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ see below for the correct formula.

3. Basic Properties (textbook p819)

(1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (Negative sign!)

(2) $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$

(3) $\vec{a} \times (\vec{b} \pm \vec{c}) = \vec{a} \times \vec{b} \pm \vec{a} \times \vec{c}$

$(\vec{a} \pm \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} \pm \vec{b} \times \vec{c}$.

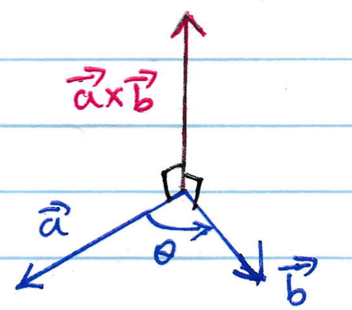
4. Advanced Properties

(1) $\underbrace{|\vec{a} \times \vec{b}|}_{\text{length of the vector } \vec{a}\times\vec{b}} = |\vec{a}|\,|\vec{b}| \sin(\theta)$.

length of the vector $\vec{a} \times \vec{b}$. Angle between $\vec{a}$ and $\vec{b}$

(2) Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel iff $\vec{a} \times \vec{b} = \vec{0}$.

(3) Right-hand rule



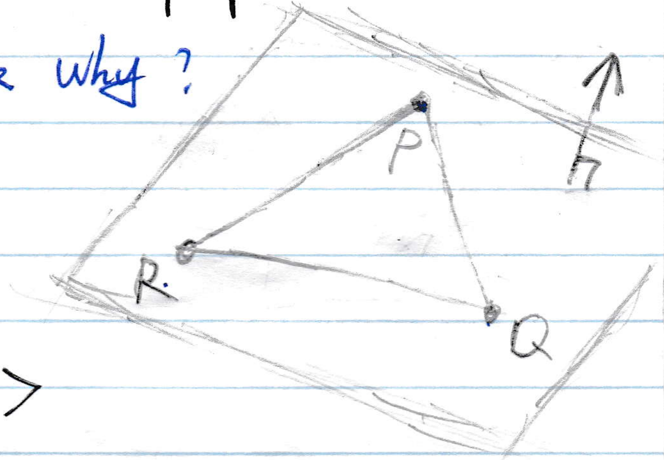
$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

(4) $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$

$\vec{j} \times \vec{i} = -\vec{k}$, $\vec{k} \times \vec{j} = -\vec{i}$, $\vec{i} \times \vec{k} = -\vec{j}$.

Text-Ex 3: Find a vector perpendicular to the plane that passes through the points $P(1,4,6)$, $Q(-2,5,-1)$ and $R(1,-1,1)$.

Ans: Equivalent to finding a vector perpendicular to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Think Why?



$$\overrightarrow{PQ} = \langle -2-1, 5-4, -1-6 \rangle = \langle -3, 1, -7 \rangle$$

$$\overrightarrow{PR} = \langle 1-1, -1-4, 1-6 \rangle = \langle 0, -5, -5 \rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -7 \\ -5 & -5 \end{vmatrix} \vec{i} - \begin{vmatrix} -3 & -7 \\ 0 & -5 \end{vmatrix} \vec{j} + \begin{vmatrix} -3 & 1 \\ 0 & -5 \end{vmatrix} \vec{k}$$

$$= \big(1(-5)-(-7)(-5)\big)\vec{i} - \big((-3)(-5)-(-7)(0)\big)\vec{j} + \big((-3)(-5)-1(0)\big)\vec{k}$$

$$= -40\vec{i} - 15\vec{j} + 15\vec{k}$$

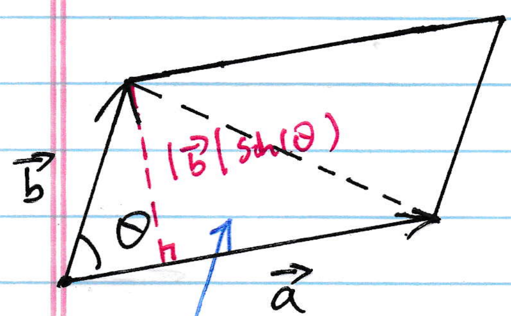
5. (Scalar) Triple Products:

the answer is a number not a vector

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

where $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, $\vec{c} = \langle c_1, c_2, c_3 \rangle$.

6. Properties Related to Geometry

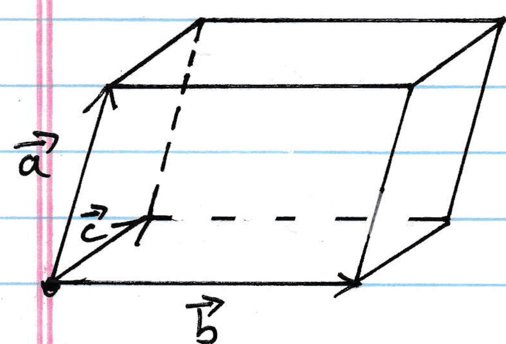


Area of the parallelogram determined by $\vec{a}$ and $\vec{b}$

$$= |\vec{a}|\,|\vec{b}|\sin(\theta) = |\vec{a}\times\vec{b}|$$

$\theta \in [0, \pi]$

If we consider the area of the $\triangle$, then $\text{Area}_{\triangle} = \frac{1}{2}|\vec{a}\times\vec{b}|$.



The volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$= |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

Text-Ex 4: Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$ and $R(1,-1,1)$.

Ans: By the observation above,

$$\text{Area}_{\triangle PQR} = |\overrightarrow{PQ}\times\overrightarrow{PR}|\cdot\frac{1}{2}$$

(Area of the triangle is half the area of the parallelogram)

From the calculations for Text-Ex 3,

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \langle -40, -15, 15\rangle,$$

so $$|\overrightarrow{PQ}\times\overrightarrow{PR}| = \sqrt{(-40)^2+(-15)^2+15^2} = \sqrt{2050} = 5\sqrt{82}$$

$\Rightarrow$ Area of $\triangle PQR$ is $\frac{5}{2}\sqrt{82}$.

Text-Exercise 35 : Find the volume of the parallelepiped with adjacent edges PQ, PR and PS where P(-2,1,0), Q(2,3,2), R(1,4,-1), S(3,6,1).

Ans : Using the observation,

$$\text{Volume} = \left| \vec{PQ} \cdot \left(\vec{PR} \times \vec{PS} \right) \right|$$

Since $\vec{PQ} = \langle 4, 2, 2 \rangle$,

$\vec{PR} = \langle 3, 3, -1 \rangle$

$\vec{PS} = \langle 5, 5, 1 \rangle$,

by formula for triple products, determinant of order 3

$$\vec{PQ} \cdot (\vec{PR} \times \vec{PS}) = \begin{vmatrix} 4 & 2 & 2 \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix}$$

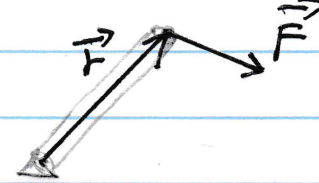
$$= 4 \begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} + 2 \begin{vmatrix} 3 & 3 \\ 5 & 5 \end{vmatrix}$$

$$= 4\left(3(1) - (-1)5\right) - 2\left(3(1) - (-1)5\right) + 2\left(3(5) - 3(5)\right)$$

$$= 4(8) - 2(8) + 2(0) = 16$$

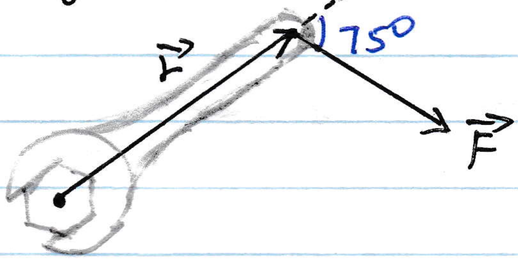
Therefore, the volume is 16.

(Unimportant) 7. Application in Physics : Torque



(torque) $\vec{\tau} = \vec{r} \times \vec{F}$

Text-Ex 6 : A bolt is tightened by applying a 40-N force $\vec{F}$ to a 0.25-m wrench. The angle between $\vec{F}$ and $\vec{r}$ is 75°. Find the magnitude of the torque about the center of the bolt.



Ans: We want to compute $|\vec{\tau}|$ and we know

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

So
$$|\vec{\tau}| = |\vec{r}|\,|\vec{F}|\sin(\theta)$$
$$= (0.25)(40)\sin(75°) = 10\sin(75°)$$
$$\approx 9.66 \text{ N}\cdot\text{m}$$

8. Other Advanced Properties (~~Not required~~ although these are from textbook p 819)

(1) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

(2) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.