Ch8 Series Solutions of Differential Equations

8.1 Introduction: Thy Taylor Polynomial Approximation

- Review of Taylor expansion :

Taylor polynomial of degree n centered at x_0 for $f(x)$ is:

$$\varphi_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$= \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$

where we do the Taylor expansion.

$P_n(x)$ is a good approximation of $f(x)$ for x around x_0.

Example: Determine the Taylor polynomial of degree x_0 for function $f(x)$ at $x = x_0$.

(a) $f(x) = e^x$, $n = 4$, $x_0 = 0$

Solution: To find the Taylor polynomial, we need to find $f(x_0)$, $f'(x_0)$, $f''(x_0)$, $f'''(x_0)$, $f''''(x_0)$ first.

$$f(x_0) = e^0 = 1$$

$$f'(x) = e^x, \quad f'(x_0) = e^0 = 1$$

Also, $f''(x) = f'''(x) = f''''(x) = e^x$, $f''(x_0) = f'''(x_0) = f''''(x_0) = 1$.

So by formula, the taylor polynomial is

$$\varphi_4(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4.$$

(b) $f(x) = e^x$, $n = 4$, $x_0 = 2$

Solution: Almost the same as part (a), but we are doing Taylor expansion at a different point $x = 2$.

Similar to (a), $f'(x) = f''(x) = f'''(x) = f''''(x) = e^x$

So

$$f'(x_0) = f''(x_0) = f'''(x_0) = f''''(x_0) = e^{x_0} = e^2.$$

By formula,

$$P_4(x) = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \frac{e^2}{4!}(x-2)^4$$

$$f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f''''(x_0)}{4!}(x-x_0)^4$$

(c) $f(x) = \ln(x)$, $n = 3$, $x_0 = 1$

Solution: $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = (-x^{-2})' = 2x^{-3} = \frac{2}{x^3}$

So:

$$f(x_0) = \ln(1) = 0$$

$$f'(x_0) = \frac{1}{1} = 1, \quad f''(x_0) = -\frac{1}{1^2} = -1, \quad f'''(x_0) = \frac{2}{1^3} = 2.$$

By formula,

$$P_3(x) = 0 + 1\cdot(x-x_0) + \frac{(-1)}{2!}(x-x_0)^2 + \frac{2}{3!}(x-x_0)^3$$

$$= 0 + (x-1) + \frac{-1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

In the above examples, expression of $y(x)$ is known so that the Taylor expansion can be computed easily.

Next, we show how to compute Taylor expansion of $y(x)$, which is the unknown solution to IVP.

Example: Find fifth-degree Taylor polynomial around $x_0 = 0$ for the solution of IVP:

$$y'' = 3y' + x^2 y$$

$$y(0) = 10, \quad y'(0) = 5.$$

Solution: To construct

$$P_5(x) = y(0) + y'(0)\,x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5,$$

we need values of $y(0)$, $y'(0)$, $y''(0)$, $y'''(0)$, $y^{(4)}(0)$, $y^{(5)}(0)$.

We have already known $y(0) = 10$, $y'(0) = 5$.

What about $y''(0)$? Since $y'' = 3y' + x^2y$,

$$y''(0) = 3y'(0) + 0^2 \cdot y(0) = 3 \cdot 5 + 0 \cdot 10 = 15$$

$y'''(0)$? Take derivatives on both sides of $y'' = 3y' + x^2y$

$$\frac{d}{dx}(y'') = \frac{d}{dx}(3y' + x^2y)$$

$$y''' = 3y'' + (2xy + x^2y') \qquad (\Delta_1)$$

$$\Rightarrow \quad y'''(0) = 3y''(0) + \underbrace{2 \cdot 0 \cdot y(0) + 0^2 \cdot y'(0)}_{=0} = 3 \cdot 15 = 45$$

Continue for $y^{(4)}(0)$. Take derivatives of (Δ_1):

$$\frac{d}{dx}(y''') = \frac{d}{dx}(3y'' + 2xy + x^2y')$$

(Attention! We are not taking derivatives of $y'''(0) = 45$, but we are taking derivatives for functions (eq of functions)

$$y''' = 3y'' + 2xy + x^2y'\ )$$

So:

$$y^{(4)} = 3y''' + (2y + 2xy') + (2xy' + x^2y'')$$

$$y^{(4)} = 3y''' + 2y + 4xy' + x^2y'' \qquad (\Delta_2)$$

$$\Rightarrow \quad y^{(4)}(0) = 3y'''(0) + 2y(0) + 4 \cdot 0 \cdot y'(0) + 0^2 \cdot y''(0)$$

$$= 3 \cdot 45 + 2 \cdot 10 + 0 + 0 = 155$$

Continue for $y^{(5)}(0)$. Take derivatives of (Δ_2):

$$\frac{d}{dx}(y^{(4)}) = \frac{d}{dx}(3y''' + 2y + 4xy' + x^2y'')$$

$$y^{(5)} = 3y^{(4)} + 2y' + (4y' + 4xy'') + (2xy'' + x^2y''')$$
$$= 3y^{(4)} + 6y' + 6xy'' + x^2y'''$$
$$\Rightarrow y^{(5)}(0) = 3\cdot 155 + 6\cdot 5 + 6\cdot 0\cdot y''(0) + 0^2\cdot y'''(0) = 495.$$

Therefore

$$P_5(x) = 10 + 5x + \frac{15}{2!}x^2 + \frac{45}{3!}x^3 + \frac{155}{4!}x^4 + \frac{495}{5!}x^5.$$

<u>Example</u>: Determine the Taylor polynomial of degree 3 for the solution to the IVP (at $x=0$): $y' = \frac{1}{x+y+1}$, $y(0) = 0$.

<u>Solution</u>: To construct

$$P_3(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3,$$

we need $y(0)$, $y'(0)$, $y''(0)$, $y'''(0)$. We've already known $y(0) = 0$. To get $y'(0)$, plug in or letting $x=0$ in

$$y' = \frac{1}{x+y+1}$$

$$\Rightarrow y'(0) = \frac{1}{0+y(0)+1} = \frac{1}{1} = 1.$$

$y''(0)$? Take derivatives of $y' = \frac{1}{x+y+1}$.

$$\frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{1}{x+y+1}\right)$$

$$y'' = -\frac{1}{(x+y+1)^2}\cdot\frac{d}{dx}(x+y+1) \leftarrow \text{Chain rule!}$$

$$y'' = -\frac{1}{(x+y+1)^2}\cdot(1+y'+0)$$

$$\Rightarrow y''(0) = -\frac{1}{(0+\underbrace{y(0)}_{=0}+1)^2}\cdot(1+1+0) = -2.$$

$$\frac{d}{dx}(y'') = \frac{d}{dx}\left(-\frac{1}{(x+y+1)^2}\cdot(1+y')\right)$$

$$y''' = -\frac{(-2)}{(x+y+1)^3}\cdot\frac{d}{dx}(x+y+1)\cdot(1+y')$$

$$+\left(-\frac{1}{(x+y+1)^2}\cdot\frac{d}{dx}(1+y')\right)$$

Product rule & chain rule

$$= \frac{2}{(x+y+1)^3}\cdot(1+y'+0)\cdot(1+y')$$

$$-\frac{1}{(x+y+1)^2}\cdot y''$$

$$= \frac{2}{(x+y+1)^3}\cdot(y'+1)^2 - \frac{1}{(x+y+1)^2}\cdot y''$$

$$\Rightarrow y'''(0) = \frac{2}{(0+y(0)+1)^3}\cdot(y'(0)+1)^2 - \frac{1}{(0+y(0)+1)^2}\cdot y''(0)$$

$$= \frac{2}{1^3}\cdot(1+1)^2 - \frac{1}{1^2}\cdot(-2) = 10.$$

Therefore $\varphi_3(x) = 0 + 1\cdot x + \frac{-2}{2!}x^2 + \frac{10}{3!}x^3.$

8.2 Power Series and Analytic Functions

Power Series: $\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$

We say it converges at $x=c$ if $\sum_{n=0}^{\infty} a_n(c-x_0)^n$ converges.

Radius of Convergence: ρ

Power series converges for $|x-x_0| < \rho$ and diverges for $|x-x_0| > \rho$.

Not the main focus of this class

Ratio Test: If $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = L$, $(0 \le L \le +\infty)$

then radius of convergence $\rho = L$.

Product of two power series:

Given $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, $g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n$,

then $f(x)\,g(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n$ where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Example: $f(x) = \sum_{n=0}^{\infty} n x^n = x + 2x^2 + 3x^3 + \cdots$

$g(x) = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$

Compute the first few terms (until x^4) of $f(x)\,g(x)$

Solution: $(x + 2x^2 + 3x^3 + 4x^4 + \cdots)\underbrace{(1 - x + x^2 - x^3 + \cdots)}$

$= x + (-x^2 + 2x^2)$

$+ (3x^3 - 2x^2 \cdot x + x \cdot x^2)$

$+ (4x^4 - 3x^3 \cdot x + 2x^2 \cdot x^2 - x \cdot x^3)$

$+ \cdots$

$= x + x^2 + 2x^3 + 2x^4 + \cdots$

Here we only write until x^3 because x^4 will only give us terms after x^5 since $f(x) = x + \cdots$

Example: Compute $(1+x^2)\sum_{n=0}^{+\infty} a_n x^n$

Solution: $(1+x^2)\sum_{n=0}^{+\infty} a_n x^n$

$= \sum_{n=0}^{+\infty} a_n x^n + x^2 \sum_{n=0}^{+\infty} a_n x^n$

$= \underline{\sum_{n=0}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} a_n x^{n+2}}$

We'll learn how to combine these terms later

Differentiation and Integration of Power Series

If $f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n$ has a radius of convergence $\rho > 0$. Then

$$f'(x) = \sum_{n=1}^{+\infty} n a_n (x-x_0)^{n-1}, \qquad |x-x_0| < \rho$$

and

$$\int f(x)\,dx = \sum_{n=0}^{+\infty} \frac{a_n}{n+1} (x-x_0)^{n+1} + C, \qquad |x-x_0| < \rho.$$

$\hookrightarrow$ It seems that we do not need this in our class.

One comment: this formula of $f'(x)$ is obtained just by taking derivatives of each term in the series for $f(x)$.

Note $\frac{d}{dx}\left(a_n (x-x_0)^n\right) = a_n \cdot n (x-x_0)^{n-1}$.

Example Compute $f'(x)$ where

$$f(x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$$

Solution:

$$f'(x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \cdot n (x-1)^{n-1}$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} (x-1)^{n-1}.$$

Shifting the Summation Index

First, in $\sum_{n=0}^{+\infty} a_n (x-x_0)^n$, the variable/index n is a dummy index, i.e. $\sum_{n=0}^{+\infty} a_n (x-x_0)^n = \sum_{k=0}^{+\infty} a_k (x-x_0)^k$.

But we need to do things more complicated than this.

Example: Express $\sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2}$ as a series with term x^k instead of x^{n-2}.

Solution: What we do here is a change of variables for Σ. Let $k=n-2$, then $n=k+2$. Then the limits $n=2$ to $+\infty$ would become $k=0$ to $+\infty$

so:
$$\sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2} = \sum_{k=0}^{+\infty} (k+2)(k+2-1) a_{k+2} x^k$$
$$= \sum_{k=0}^{+\infty} (k+2)(k+1) a_{k+2} x^k$$

Example: Compute $(1+x^2)\sum_{n=0}^{+\infty} a_n x^n$

Solution: $(1+x^2)\sum_{n=0}^{+\infty} a_n x^n$
$$= \sum_{n=0}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} a_n x^{n+2}$$
$$= \sum_{n=0}^{+\infty} a_n x^n + \sum_{k=2}^{+\infty} a_{k-2} x^k \qquad (k=n+2)$$
$$= \sum_{n=0}^{+\infty} a_n x^n + \sum_{n=2}^{+\infty} a_{n-2} x^n$$
$$= a_0 + a_1 x + \underbrace{\sum_{n=2}^{+\infty} a_n x^n + \sum_{n=2}^{+\infty} a_{n-2} x^n}_{\text{Combine these two terms}}$$
$$= a_0 + a_1 x + \sum_{n=2}^{+\infty} (a_n + a_{n-2}) x^n$$

Example: Given $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, compute $xf(x) + f'(x)$.

Solution: $x f(x) + f'(x)$

$(k=n+1)$
$$= \sum_{n=0}^{+\infty} a_n x^{n+1} + \sum_{n=1}^{+\infty} n a_n x^{n-1} \qquad (k=n-1)$$
$$= \sum_{k=1}^{+\infty} a_{k-1} x^k + \sum_{k=0}^{+\infty} (k+1) a_{k+1} x^k$$

Since we want to combine

$$\sum_{k=1}^{+\infty} a_{k-1} x^k \quad \text{and} \quad \sum_{k=0}^{+\infty} (k+1) a_{k+1} x^k ,$$

we need to make the lower bound and upper bound (for Σ) to be the same in two Σs.

An easy way is to write

$$\sum_{k=0}^{+\infty} (k+1) a_{k+1} x^k = (0+1) a_{0+1} x^0 + \sum_{k=1}^{+\infty} (k+1) a_{k+1} x^k$$

$$= a_1 + \sum_{k=1}^{+\infty} (k+1) a_{k+1} x^k$$

So $x f(x) + f'(x)$

$$= \sum_{k=1}^{+\infty} a_{k-1} x^k + \left(a_1 + \sum_{k=1}^{+\infty} (k+1) a_{k+1} x^k\right)$$

$$= a_1 + \sum_{k=1}^{+\infty} \left(a_{k-1} + (k+1) a_{k+1}\right) x^k .$$

<u>Example</u>: Compute $x \sum_{n=0}^{+\infty} a_n (x-x_0)^n$.

<u>Solution</u>: Since we are dealing with series at x_0, write $x = \underline{(x-x_0) + x_0}$

↳ This is a series with two terms.

Then

$$x \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

$$= \left((x-x_0) + x_0\right) \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

$$= (x-x_0) \sum_{n=0}^{+\infty} a_n (x-x_0)^n + x_0 \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

$$= \sum_{n=0}^{+\infty} a_n (x-x_0)^{n+1} + \sum_{n=0}^{+\infty} x_0 a_n (x-x_0)^n$$

$$= \sum_{k=1}^{+\infty} a_{k-1} (x-x_0)^k + \sum_{k=0}^{+\infty} x_0 a_k (x-x_0)^k$$

$$= \left[\sum_{k=1}^{+\infty} \left(a_{k-1} + x_0 a_k\right) (x-x_0)^k\right] + x_0 a_0$$

8.3 Power Series Solutions to Linear Differential Equations

In this section, we'll discuss a method for obtaining a power series solution easier than the Taylor series method in section 8.1.

Example: Find a power series solution about $x=0$ to

$$y' + 2xy = 0$$

Solution: Write

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

and our goal is to determine the coefficients a_n.

Since

$$y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1},$$

we have

$$\begin{aligned}
0 &= y' + 2xy \\
&= \sum_{n=1}^{+\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{+\infty} a_n x^n \\
(k=n-1) \quad &= \sum_{k=0}^{+\infty} (k+1) a_{k+1} x^k + \sum_{n=0}^{+\infty} 2a_n x^{n+1} \\
&= \sum_{k=0}^{+\infty} (k+1) a_{k+1} x^k + \sum_{k=1}^{+\infty} 2a_{k-1} x^k \qquad (*) \\
&= (0+1) a_{0+1} x^0 + \sum_{k=1}^{+\infty} (k+1) a_{k+1} x^k + \sum_{k=1}^{+\infty} 2a_{k-1} x^k \\
&= a_1 + \sum_{k=1}^{+\infty} \left(a_{k+1}(k+1) + 2a_{k-1} \right) x^k
\end{aligned}$$

$$\Rightarrow \quad a_1 = 0.$$

$$a_{k+1} \cdot (k+1) + 2a_{k-1} = 0, \quad k \geq 1.$$

The second line above gives us

$$a_{k+1} = -\frac{2}{k+1} a_{k-1} \quad \leftarrow \text{recurrence relation}$$

Setting $k=1,2,\cdots,8$ and using $a_1=0$, we find

$$a_2 = -\frac{2}{2} = -a_0 \ (k=1), \quad a_3 = -\frac{2}{3}a_1 = 0 \ (k=2)$$

$$a_4 = -\frac{2}{4}a_2 = \frac{1}{2}a_0 \ (k=3), \quad a_5 = -\frac{2}{5}a_3 = 0 \ (k=4)$$

we also use $a_2=-a_0$ here

$$a_6 = -\frac{2}{6}a_4 = -\frac{1}{3!}a_0 \ (k=5), \quad a_7 = -\frac{2}{7}a_5 = 0 \ (k=6)$$

$$a_8 = -\frac{2}{8}a_6 = \frac{1}{4!}a_0 \ (k=7), \quad a_9 = -\frac{2}{9}a_7 = 0 \ (k=8)$$

Now you may ask what a_0 is. The answer is we don't know. Why? Because we are only given an ODE without initial values. If we know what $y(0)$ is, then $\underline{a_0 = y(0)}$.

b/c $y(0) = a_0 + a_1\cdot 0 + a_2 \cdot 0^2 + \cdots = a_0$.

From the calculation of a_k $(k=1,\cdots,8)$, we realize that

$$\left.\begin{aligned} a_{2n} &= \frac{(-1)^n}{n!}a_0, \quad n=1,2,\cdots \\ a_{2n+1} &= 0, \quad n=1,2,\cdots \end{aligned}\right\}$$

We won't require solving very difficult problems for a_n (any n).

$$\Rightarrow y(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!}a_0 x^{2n}$$

<u>Remark</u>: If you have difficulty in understanding $(*)$, first compute $y'+2xy$ by writing out the first few terms.

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots\cdots$$

$$2xy(x) = 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + \cdots\cdots$$

so: $0 = y'+2xy = a_1 + (2a_2+2a_0)x + (3a_3+2a_1)x^2 + (4a_4+2a_2)x^3 + \cdots\cdots$

Group term with same x^k together.

This tells us:

$$a_1 = 0 \ , \ 2a_2 + 2a_0 = 0 \ , \ 3a_3 + 2a_1 = 0 \ , \ 4a_4 + 2a_2 = 0 \ , \ \cdots$$

and thus we can obtain

$$a_1 = a_3 = 0 \ , \quad a_2 = -a_0 \ , \quad a_4 = -\tfrac{1}{2}a_2 = \tfrac{1}{2}a_0$$

In this way, we can obtain the first few terms in the power series solution, but it is not as powerful as (*) in finding the general expression of a_n.

General Procedure of Finding a Power Series Solution

1. Write $y(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$

   depends on where you are asked to find power series solution, often $x_0 = 0$

2. Plug into the ODE and rewrite it as

   $$0 = \sum (\quad) x^n + (\text{maybe something like } a_1 \ , \ a_1 x + a_0 \ \cdots)$$

   Only has one single Σ

3. Derive the recurrence relation

4. Compute the first few terms or the expression for any a_n.

Example: Find the solution to $y'' = xy$ in the form of a power series about the point $x = 0$.

Solution: Write $y(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n = \sum_{n=0}^{+\infty} a_n x^n$.

(with $x_0 = 0$)

In order to plug this into the ODE, we compute

$$y' = \sum_{n=1}^{+\infty} n a_n x^{n-1} \ , \quad y'' = \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2}$$

So
$$\begin{aligned} y'' - xy &= \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{+\infty} a_n x^n \\ &= \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{+\infty} a_n x^{n+1} \end{aligned}$$

$$= \sum_{k=0}^{+\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{+\infty} a_{k-1} x^k$$

$$= (0+2)(0+1) a_{0+2} + \sum_{k=1}^{+\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{+\infty} a_{k-1} x^k$$

$$= 2a_2 + \sum_{k=1}^{+\infty} \left[(k+2)(k+1) a_{k+2} - a_{k-1} \right] x^k$$

$$\Rightarrow \quad 0 = 2a_2$$

$$0 = (k+2)(k+1) a_{k+2} - a_{k-1}, \quad k \geq 1$$

So $a_2 = 0$; $\underbrace{a_{k+2} = \frac{1}{(k+2)(k+1)} a_{k-1}, \quad k \geq 1}_{\text{recurrence relation}}$

Choosing different k we will get :

$$a_3 = \frac{1}{3 \cdot 2} a_0, \quad a_4 = \frac{1}{4 \cdot 3} a_1$$

$$a_5 = \frac{1}{5 \cdot 4} a_2 = 0, \quad a_6 = \frac{1}{6 \cdot 5} a_3 = \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} a_0$$

$$a_7 = \frac{1}{7 \cdot 6} a_4 = \frac{1}{7 \cdot 6 \cdot 4 \cdot 3} a_1,$$

$$a_8 = \frac{1}{8 \cdot 7} a_5 = \frac{1}{8 \cdot 7 \cdot 5 \cdot 4} a_2 = 0.$$

So $a_{3n+2} = \frac{1}{(3n+2)(3n+1)} a_{3(n-1)+2} = \cdots = \frac{1}{(3n+2)(3n+1)(3n-1)(3n-2) \cdots 5 \cdot 4} a_2$

$$= 0$$

$$a_{3n+1} = \frac{a_{3(n-1)+1}}{(3n+1)(3n)} = \cdots = \frac{a_1}{(3n+1)(3n)\underbrace{(3n-2)}_{3(n-1)+1}\underbrace{(3n-3)}_{3(n-1)} \cdots 4 \cdot 3}$$

$$a_{3n} = \frac{a_{3(n-1)}}{(3n)(3n-1)} = \cdots = \frac{a_0}{(3n)(3n-1)\underbrace{(3n-3)}_{3(n-1)}\underbrace{(3n-4)}_{3(n-1)-1} \cdots 3 \cdot 2}$$

This gives us

$$y = a_0 + \sum_{n=1}^{+\infty} \frac{a_0}{(3n)(3n-1)(3n-3)(3n-4)\cdots 3\cdot 2} x^{3n}$$

$$+ a_1 x + \sum_{n=1}^{+\infty} \frac{a_1}{(3n+1)(3n)(3n-2)(3n-3)\cdots 4\cdot 3} x^{3n+1}$$

☆ The exam problems will be similar in the sense that they will ask for the recurrence relation and several terms in the power series.

Example: Find the recurrence relation and compute the first five non-zero terms in the power series solution to (at $x_0 = 0$)

$$2y'' + xy' + y = 0$$

$$y(0) = y_0 \neq 0, \quad y'(0) = y_1 \neq 0.$$

Solution: Write $y(x) = \sum_{n=0}^{+\infty} a_n x^n$, then

$$y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2}.$$

Plugging the above into our ODE we get:

$$2y'' + xy' + y$$

$$= \sum_{n=2}^{+\infty} 2n(n-1) a_n x^{n-2} + \sum_{n=1}^{+\infty} x \cdot n a_n x^{n-1} + \sum_{n=0}^{+\infty} a_n x^n$$

$$= \sum_{k=0}^{+\infty} 2(k+2)(k+2-1) a_{k+2} x^k + \sum_{n=1}^{+\infty} n a_n x^n + \sum_{n=0}^{+\infty} a_n x^n$$

(Change of variable, $k = n-2$)

$$= \sum_{n=0}^{+\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{+\infty} n a_n x^n + \sum_{n=0}^{+\infty} a_n x^n$$

We can change from k to n b/c this index is dummy.
Notice we've changed all the powers to be x^n in the "Σ".

$$= 2(0+2)(0+1) a_{0+2} + a_0 + \sum_{n=1}^{+\infty} \left(2(n+2)(n+1) a_{n+2} + (n+1) a_n\right) x^n$$

$$= 4a_2 + a_0 + \sum_{n=1}^{+\infty} \left(2(n+2)(n+1) a_{n+2} + (n+1) a_n\right) x^n$$

Since we know $2y'' + xy' + y$, the equation above implies that:

$$\begin{cases} 4a_2 + a_0 = 0 \\ 2(n+2)(n+1)a_{n+2} + (n+1)a_n = 0 \quad , \quad n \geq 1 \end{cases}$$

The second line gives us the recurrence relation:

$$\boxed{a_{n+2} = \frac{-(n+1)a_n}{2(n+2)(n+1)} = \frac{-1}{2(n+2)} a_n \quad , \quad n \geq 1}.$$

We also get $a_2 = -\frac{1}{4}a_0$ from $4a_2 + a_0 = 0$.

Taking different n in the recurrence relation gives

$$a_3 = \frac{-1}{6}a_1 \quad , \quad a_4 = \frac{-1}{8}a_2 = \frac{1}{32}a_0$$

$$a_5 = \frac{-1}{10}a_3 = \frac{-1}{10} \cdot \frac{-1}{6}a_1 = \frac{1}{60}a_1 \quad , \cdots$$

If we are given initial values for the ODE, we can use them to determine some coefficients in the power series.

From $y(x) = \sum_{n=0}^{+\infty} a_n x^n$, we see $y(0) = a_0$;

From $y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$, we see $y'(0) = 1 \cdot a_1 = a_1$

$$\Rightarrow \quad a_0 = y_0 \neq 0 \quad , \quad a_1 = y_1 \neq 0$$

Therefore $\quad a_0 = y_0 \quad , \quad a_1 = y_1 \quad ,$

$$a_2 = -\frac{1}{4}y_0 \quad , \quad a_3 = -\frac{1}{6}y_1 \quad , \quad a_4 = \frac{1}{32}y_0$$

The first five non-zero terms in the power series are:

$$y(x) = y_0 + y_1 x + \left(-\frac{1}{4}y_0\right)x^2 + \left(-\frac{1}{6}y_1\right)x^3 + \left(\frac{1}{32}y_0\right)x^4 + \cdots\cdots$$

In the next example we consider a power series at $x_0 \neq 0$.

<u>Example</u>: Find the recurrence relation and compute the first four non-zero terms in a power series expansion about $x_0 = 1$ for the solution to the following IVP:

$$y'' - xy = 0$$

$$y(1) = 1, \quad y'(1) = 2$$

<u>Solution</u>: Write $y(x) = \sum_{n=0}^{+\infty} a_n (x-1)^n$ (b/c $x_0 = 1$),

then $y'(x) = \sum_{n=1}^{+\infty} n a_n (x-1)^{n-1}$, $y''(x) = \sum_{n=2}^{+\infty} n(n-1) a_n (x-1)^{n-2}$.

Plugging the above into our ODE we get:

$$y'' - xy$$

$$= \sum_{n=2}^{+\infty} n(n-1) a_n (x-1)^{n-2} - \underbrace{\big((x-1)+1\big)}\sum_{n=0}^{+\infty} a_n (x-1)^n$$

The reason of writing $x = (x-1)+1$ is we are considering power series at $x_0 = 1$, and $(x-1)+1$ can be thought as a power series for x at $x_0 = 1$.

$$= \sum_{n=2}^{+\infty} n(n-1) a_n (x-1)^{n-2} - (x-1)\sum_{n=0}^{+\infty} a_n (x-1)^n - 1 \cdot \sum_{n=0}^{+\infty} a_n (x-1)^n$$

$$= \sum_{n=2}^{+\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=0}^{+\infty} a_n (x-1)^{n+1} - \sum_{n=0}^{+\infty} a_n (x-1)^n$$

Change of variables for "Σ"

$$= \sum_{k=0}^{+\infty} (k+2)(k+2-1) a_{k+2} (x-1)^k - \sum_{k=1}^{+\infty} a_{k-1} (x-1)^k - \sum_{k=0}^{+\infty} a_k (x-1)^k$$

$$= (0+2)(0+2-1) a_{0+2} (x-1)^0 - a_0 (x-1)^0$$

$$+ \sum_{k=1}^{+\infty} \Big((k+2)(k+1) a_{k+2} - a_{k-1} - a_k \Big) (x-1)^k = 0.$$

This gives us $\begin{cases} 2a_2 - a_0 = 0 \\ (k+2)(k+1)\,a_{k+2} - a_{k-1} - a_k = 0, & k \geq 1 \end{cases}$

So the recurrence relation is (from the second line)

$$a_{k+2} = \frac{1}{(k+2)(k+1)}(a_k + a_{k-1}), \quad k \geq 1$$

(You may also say $(k+2)(k+1)\,a_{k+2} - a_{k-1} - a_k = 0$ is the recurrence relation).

We also have $a_2 = \frac{1}{2}a_0$.

From the initial values and power series of $y(x)$, $y'(x)$, we get:

$$1 = y(1) = a_0$$

$$2 = y'(1) = a_1$$

and thus $a_2 = \frac{1}{2}a_0 = \frac{1}{2}$.

From the recurrence relation,

$$a_3 = \frac{1}{3 \cdot 2}(a_0 + a_1) = \frac{1}{6}(1+2) = \frac{1}{2}$$

$$a_4 = \frac{1}{4 \cdot 3}(a_1 + a_2) = \frac{1}{12}\left(2 + \frac{1}{2}\right) = \frac{5}{24}$$

$$\vdots \qquad \vdots$$

The first four non-zero terms can be found as:

$$y(x) = 1 + 2(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \cdots$$

Actually we don't need a_4 in this problem.