Ch 7 $\boxed{\text{Laplace Transform}}$

$\rightarrow$ A way to change a function of t to another function of s.

This chapter will discuss another method called "Laplace transform Method" which can be used to solve ODEs with constant coefficients.

Example of ODEs with constant coefficients:

$$y' = 3$$

$$y'' - 2y' - 3y = 0$$

$\Leftarrow$ We have learned how to solve these equations before, but the method in this chapter would be different.

Just to give you an idea, no need to understand right now, the procedure of using Laplace transform to solve equations look like the following:

IVP: $\quad y' = 3, \quad y(0) = 1$

$\downarrow$ Laplace Transform (Step 1)

$$\mathcal{L}[y'] = \mathcal{L}[3]$$

$$s\mathcal{L}[y] - y(0) = \frac{3}{s}$$

$\downarrow$ Algebraic calculation (Step 2)

$$\mathcal{L}[y] = \frac{3}{s^2} + \frac{1}{s}$$

$\downarrow$ Inverse Laplace Transform (Step 3)

$$y(t) = 3t + 1$$

No need to understand the mathematics above. The message I want to convey here is that the solving process contains three steps. We'll discuss how to do the Laplace transform and inverse Laplace transform in this chapter.

7.2 Definition of the Laplace Transform

__Definition__: Let $f(t)$ be a function on $(0, \infty)$. the Laplace transform of $f(t)$ is

$$\mathcal{L}[f(t)](s) := F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

The domain of $F(s)$ is all value of s for which the integral exists.

We usually use capital F to denote the Laplace transform of $f(t)$, and similarly Y for the Laplace transform of $y(t)$. No reasons. Just notation and convention.

__Example__: $f(t) = 1$, then

$$\begin{aligned}
\mathcal{L}[f(t)](s) &= \int_0^\infty e^{-st} f(t)\,dt \\
&= \int_0^\infty e^{-st}\,dt \\
&= \lim_{N\to\infty} \int_0^N e^{-st}\,dt \\
&= \lim_{N\to\infty} \left(-\frac{1}{s} e^{-st}\right)\Big|_{t=0}^{N} \\
&= \lim_{N\to\infty} \left(-\frac{1}{s} e^{-sN} + \frac{1}{s} e^{-s\cdot 0}\right) \\
&= 0 + \frac{1}{s} = \frac{1}{s} \qquad (s > 0)
\end{aligned}$$

(You can also write as $\mathcal{L}[f](s)$)

We need this b/c in the calculation we use $\lim_{N\to\infty} e^{-sN} = 0$.

This tells us: $\mathcal{L}[1] = \frac{1}{s}$ $(s>0)$.

domain of $F(s)$.

Example: $f(t) = t$, then

$$\mathcal{L}[f(t)](s) = \int_0^\infty t e^{-st}\, dt$$

$$= \lim_{N\to\infty} \int_0^N t e^{-st}\, dt$$

b/c we want to do integration by parts

$$= \lim_{N\to\infty} \int_0^N t \underline{\frac{d}{dt}\left(-\frac{1}{s} e^{-st}\right)}\, dt$$

$$= \lim_{N\to\infty} \left(t\left(-\frac{1}{s}\right)e^{-st}\Big|_{t=0}^{N} - \int_0^N \frac{d}{dt}(t)\cdot\left(-\frac{1}{s}e^{-st}\right)dt\right)$$

$$= \lim_{N\to\infty}\left(N\cdot\left(-\frac{1}{s}\right)e^{-sN} - \int_0^N -\frac{1}{s}e^{-st}\,dt\right)$$

b/c $\lim_{N\to\infty} N\cdot\left(-\frac{1}{s}\right)e^{-sN} = 0$ for $s>0$.

$$= \lim_{N\to\infty} \int_0^N \frac{1}{s} e^{-st}\, dt$$

$$= \lim_{N\to\infty} \frac{1}{s}\cdot\left(-\frac{1}{s}\right)e^{-st}\Big|_{t=0}^{N}$$

$$= \lim_{N\to\infty}\left(\frac{1}{s}\left(-\frac{1}{s}\right)e^{-sN} - \frac{1}{s}\left(-\frac{1}{s}\right)e^{-s\cdot 0}\right)$$

$$= 0 - \frac{1}{s}\cdot\left(-\frac{1}{s}\right) = \frac{1}{s^2}, \quad s>0$$

This tells us $\mathcal{L}[t] = \frac{1}{s^2}$ $(s>0)$.

Remark: For simplicity, we will write

$$t\left(-\frac{1}{s}\right)e^{-st}\Big|_{t=0}^{\infty} \text{ instead of } \lim_{N\to\infty} t\left(-\frac{1}{s}\right)e^{-st}\Big|_{t=0}^{N}.$$

They mean the same thing, but avoiding writing lim is convenient sometimes.

Example: $f(t) = e^{\alpha t}$ where α is a constant, then

$$\mathcal{L}[f(t)](s) = \int_0^\infty e^{-st} \cdot e^{\alpha t}\, dt$$

$$= \int_0^\infty e^{(\alpha - s)t}\, dt$$

$$= \frac{1}{\alpha - s} e^{(\alpha-s)t} \Big|_{t=0}^{\infty}$$

If $s > \alpha$, then $\alpha < s$ and $\lim_{N\to\infty} \frac{1}{\alpha - s} e^{(\alpha-s)N} = 0$

$$= \left(\lim_{N\to\infty} \frac{1}{\alpha - s} e^{(\alpha-s)N} \right) - \frac{1}{\alpha - s}$$

$$= 0 - \frac{1}{\alpha - s} = \frac{1}{s-\alpha} \qquad (s > \alpha)$$

This tells us $\mathcal{L}[e^{\alpha t}] = \frac{1}{s-\alpha} \quad (s > \alpha)$.

Example: $f(t) = \sin(bt)$, where $b \neq 0$ is a constant, then

$$\mathcal{L}[f(t)](s) = \int_0^\infty e^{-st} \sin(bt)\, dt$$

Using integral table for $\int e^{-st} \sin(bt)\, dt$

$$= \left(\frac{e^{-st}}{s^2+b^2} \left(-s \sin(bt) - b\cos(bt)\right) \Big|_0^\infty \right)$$

When $s > 0$, $\lim_{N\to\infty} \frac{e^{-sN}}{s^2+b^2}\left(-s\sin(bt) - b\cos(bt)\right) = 0$

$$= -\frac{e^{-s\cdot 0}}{s^2+b^2}\left(-s \sin(b\cdot 0) - b\cos(b\cdot 0)\right)$$

$$= \frac{b}{s^2+b^2} \qquad (s>0)$$

This tells us $\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2} \quad (s>0)$.

Remark: In the example above, we use integral table for $\int e^{-st} \sin(bt)\, dt$. Let me also do it by "integration by parts" as an example/review of integration by parts.

$$\int e^{-st} \sin(bt)\, dt$$

$$= \int \frac{d}{dt}\left(-\frac{1}{s} e^{-st}\right) \sin(bt)\, dt$$

$$= -\frac{1}{s} e^{-st} \sin(bt) - \int -\frac{1}{s} e^{-st} \cdot \left(b \cos(bt)\right) dt$$

$$= -\frac{1}{s} e^{-st} \sin(bt) + \frac{b}{s} \int e^{-st} \cos(bt)\, dt$$

(We haven't got the answer, but let's continue to do integration by parts for $\int e^{-st} \cos(bt)\, dt$)

$$= -\frac{1}{s} e^{-st} \sin(bt) + \frac{b}{s} \cdot \left(\int \frac{d}{dt}\left(-\frac{1}{s} e^{-st}\right) \cos(bt)\, dt \right)$$

$$= -\frac{1}{s} e^{-st} \sin(bt) + \frac{b}{s}\left(-\frac{1}{s} e^{-st} \cos(bt) - \int -\frac{1}{s} e^{-st} \left(-b \sin(bt)\right) dt \right)$$

$$= -\frac{1}{s} e^{-st} \sin(bt) + \left(\frac{-b}{s^2}\right) e^{-st} \cos(bt) - \frac{b^2}{s^2} \int e^{-st} \sin(bt)\, dt$$

This gives us

$$\int e^{-st} \sin(bt)\, dt = -\frac{1}{s} e^{-st} \sin(bt) - \frac{b}{s^2} e^{-st} \cos(bt) - \frac{b^2}{s^2} \int e^{-st} \sin(bt)\, dt$$

So:

$$\left(1 + \frac{b^2}{s^2}\right) \int e^{-st} \sin(bt)\, dt = -\frac{1}{s} e^{-st} \sin(bt) - \frac{b}{s^2} e^{-st} \cos(bt).$$

Therefore

$$\int e^{-st} \sin(bt)\, dt = -\frac{s}{b^2+s^2} e^{-st} \sin(bt) - \frac{b}{b^2+s^2} e^{-st} \cos(bt)$$

$$(+C).$$

We can also compute the Laplace transform for discontinuous functions. See the example below

Example: $f(t) = \begin{cases} 2, & 0 \le t < 5 \\ 0, & 5 \le t < 10 \\ e^{4t}, & t \ge 10 \end{cases}$, then:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^5 e^{-st} \cdot 2\, dt + \int_5^{10} e^{-st} \cdot 0\, dt + \int_{10}^\infty e^{-st} \cdot e^{4t}\, dt$$

$$= 2\int_0^5 e^{-st}\, dt + \int_{10}^\infty e^{(4-s)t}\, dt$$

$$= 2 \cdot \left(-\frac{1}{s} e^{-st}\right)\Big|_{t=0}^{5} + \left(\frac{1}{4-s} e^{(4-s)t}\Big|_{t=10}^{\infty}\right)$$

$(s>4)$ $\rightarrow$ $$= 2 \cdot \left(-\frac{1}{s} e^{-5s} - \left(-\frac{1}{s}\right)\right) + \left(-\frac{1}{4-s} e^{(4-s)\cdot 10}\right)$$

$$= \frac{2}{s} - \frac{2e^{-5s}}{s} + \frac{e^{-10(s-4)}}{s-4} \qquad (s>4)$$

The following table of Laplace transform is helpful (Textbook page 356)

$f(t)$	$F(s) = \mathcal{L}[f](s)$
1	$\frac{1}{s}$, $s>0$
e^{at}	$\frac{1}{s-a}$, $s>a$
t^n, $n=1,2,\ldots$	$\frac{n!}{s^{n+1}}$, $s>0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$, $s>0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$, $s>0$
$e^{at} \cdot t^n$, $n=1,2,\ldots$	$\frac{n!}{(s-a)^{n+1}}$, $s>a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s>a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s>a$.

• Linearity of Laplace transform :

$$\mathcal{L}[f_1 + f_2](s) = \mathcal{L}[f_1](s) + \mathcal{L}[f_2](s)$$

$$\mathcal{L}[\underline{c}f](s) = \underline{c}\,\mathcal{L}[f](s)$$

c is a number here.

Using the property above and the table, we can compute some Laplace transform easily.

Example : Compute $\mathcal{L}[11 + 5e^{4t} - 6\sin(2t)]$.

$$\mathcal{L}[11 + 5e^{4t} - 6\sin(2t)](s)$$

$$= \mathcal{L}[11](s) + \mathcal{L}[5e^{4t}](s) + \mathcal{L}[-6\sin(2t)](s)$$

$$= 11 \cdot \mathcal{L}[1](s) + 5\,\mathcal{L}[e^{4t}](s) + (-6)\,\mathcal{L}[\sin(2t)](s)$$

$$= \frac{11}{s} + \frac{5}{s-4} - 6 \cdot \frac{2}{s^2+2^2}$$

$$= \frac{11}{s} + \frac{5}{s-4} - \frac{12}{s^2+4} \qquad (s > 4)$$

Let me explain the domain $\{s > 4\}$ a little bit.

The domain for $\mathcal{L}[1]$ are $s > 0$

$\mathcal{L}[e^{4t}]$ $s > 4$

$\mathcal{L}[\sin(2t)]$ $s > 0$

Consider the intersection of the three domains we get

$$\{s>0\} \cap \{s>4\} \cap \{s>0\} = \{s>4\}.$$

Example: Compute $\mathcal{L}[5t^2e^{-3t} - e^{12t}\cos(8t)]$.

$$\mathcal{L}[5t^2e^{-3t} - e^{12t}\cos(8t)](s)$$

$$= \mathcal{L}[5t^2e^{-3t}](s) + \mathcal{L}[-e^{12t}\cos(8t)](s)$$

$$= 5\,\mathcal{L}[t^2e^{-3t}](s) + (-1)\,\mathcal{L}[e^{12t}\cos(8t)](s)$$

$$= 5 \cdot \frac{2!}{(s-(-3))^{2+1}} - \frac{s-12}{(s-12)^2+8^2}$$

$$= \frac{10}{(s+3)^3} - \frac{s-12}{(s-12)^2+64}, \qquad s>12$$

The domain $\{s>12\}$ is obtained from

$$\{s>-3\} \cap \{s>12\}.$$

→ Not the focus of this course

Now a little bit theory because we cannot do Laplace transform for every function $f(t)$.

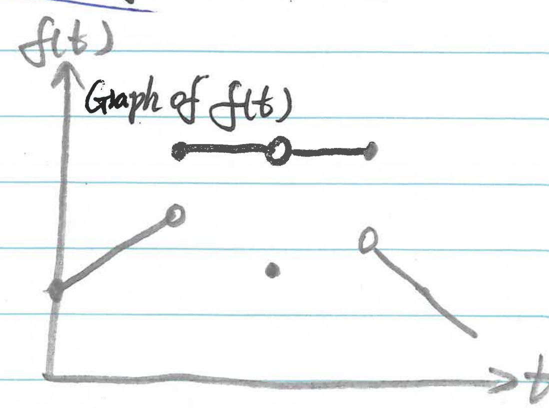
Piecewise Continuity: A function $f(t)$ is said to be piecewise continuous if it is continuous except a finite number of points (where f has a jump discontinuity).

Examples of piecewise discontinuous functions:

(1)

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2, & 1 \le t < 2 \\ (t-2)^2, & 2 \le t \le 3 \end{cases}$$

(2)



Existence of Laplace transform:

If $f(t)$ is piecewise continuous on $[0,\infty)$ and there exists $T>0$, $M>0$, $\alpha\in\mathbb{R}$ such that

$$|f(t)| \leq Me^{\alpha t} \quad \text{for all } t\geq T.$$

Then $\mathcal{L}[f](s)$ exists for $s>\alpha$.

Just as an introduction. Not Required!

Using this theorem, we can check that we can do Laplace transform for some functions. For example,

$$\left|e^{at}\sin(bt)\right| \leq 1\cdot e^{at}$$

$$\left|e^{at}\cdot t^n\right| \leq M\cdot e^{(a+\varepsilon)t} \quad \text{for all } t\geq T$$

choosing T and M large.

7.3 Properties of the Laplace Transform

- Translation in s: If $\mathcal{L}[f](s) = F(s)$, then

$$\mathcal{L}[e^{at}f(t)](s) = F(s-a).$$

Proof: Compute

$$\mathcal{L}[e^{at}f(t)](s) = \int_0^\infty e^{-st}e^{at}f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt$$

$$= F(s-a).$$

b/c $\mathcal{L}[f](s)=F(s)$ means $\int_0^\infty e^{-st}f(t)\,dt = F(s)$.

Example: Combine the property for translation in s with the Laplace transform for $\sin(bt)$:

$$\mathcal{L}[\sin(bt)](s) = \frac{b}{s^2+b^2} \longrightarrow F(s)$$

we know

$$\mathcal{L}[e^{at}\sin(bt)](s) = F(s-a) = \frac{b}{(s-a)^2+b^2}.$$

Example: Suppose we know $\mathcal{L}[\cos(bt)](s) = \frac{s}{s^2+b^2}$, compute $\mathcal{L}[e^{at}\cos(bt)]$.

Solution: According to the property for translation in s, we know

$$\mathcal{L}[e^{at}\cos(bt)](s) = F(s-a) = \frac{s-a}{(s-a)^2+b^2}.$$

- Laplace transform of the derivatives:

$$\mathcal{L}[f'](s) = s\,\mathcal{L}[f](s) - f(0)$$

Proof:

$$\mathcal{L}[f'](s) = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \lim_{N\to\infty} \int_0^N e^{-st} \underbrace{f'(t)\,dt}_{\to\, d(f(t))}$$

Integration by parts

$$= \lim_{N\to\infty} \left(e^{-st} f(t) \Big|_{t=0}^{N} - \int_0^N f(t)\, d(e^{-st}) \right)$$

$$= \lim_{N\to\infty} \left(\left(e^{-sN} f(N) - e^{-s\cdot 0} f(0)\right) - \int_0^N f(t)(-s)e^{-st}\,dt \right)$$

Consider the first part in the limit above.

$$\lim_{N\to\infty} e^{-sN} f(N) - e^{-s\cdot 0} f(0)$$

$$= 0 - f(0) = -f(0) \qquad \text{if } s > \alpha, \ |f(t)| \le Me^{\alpha t} \text{ for } t \ge T.$$

And for the second part.

$$\lim_{N\to\infty} -\int_0^N f(t)\,(-s)\,e^{-st}\,dt$$

$$= \lim_{N\to\infty} s\int_0^N f(t)\,e^{-st}\,dt = s \underbrace{\lim_{N\to\infty} \int_0^N f(t)e^{-st}\,dt}_{=\int_0^\infty f(t)e^{-st}dt = \mathcal{L}[f](s)}$$

$$= s\,\mathcal{L}[f](s)$$

Therefore combining everything together, we have

$$\mathcal{L}[f'](s) = -f(0) + s\,\mathcal{L}[f](s) = s\,\mathcal{L}[f](s) - f(0).$$

In general we have:

$$\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0)$$

$$\mathcal{L}[f'''](s) = s^3\mathcal{L}[f](s) - s^2 f(0) - sf'(0) - f''(0)$$

$$\boxed{\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)}$$

Laplace transform of high-order derivatives

Example: Use $\mathcal{L}[\sin(bt)](s) = \dfrac{b}{s^2+b^2}$ to compute $\mathcal{L}[\cos(bt)](s)$.

Solution: Let $f(t) = \sin(bt)$, then $f'(t) = b\cos(bt)$.

According to the property for Laplace transform of the derivatives,

$$\mathcal{L}[b\cos(bt)](s) = \mathcal{L}[f'](s) = s\,\mathcal{L}[f](s) - f(0)$$

$$= s \cdot \frac{b}{s^2+b^2} - \underbrace{\sin(b \cdot 0)}_{=0}$$

$$= \frac{bs}{s^2+b^2}, \quad (s>0)$$

Due to linearity of $\mathcal{L}$,

$$\mathcal{L}[b\cos(bt)](s) = b\,\mathcal{L}[\cos(bt)](s),$$

so

$$b\,\mathcal{L}[\cos(bt)](s) = \frac{bs}{s^2+b^2}, \quad (s>0)$$

$$\Longrightarrow \quad \mathcal{L}[\cos(bt)](s) = \frac{s}{s^2+b^2}, \quad (s>0).$$

- Laplace transform of $t^n f(t)$ (Derivatives of Laplace transform)

$$\mathcal{L}[t^n f(t)](s) = (-1)^n \underbrace{\frac{d^n F}{ds^n}(s)}_{\text{n-th derivative of } F(s)},$$

where $F(s) = \mathcal{L}[f(t)](s)$.

Proof: We prove for $n=1$ as an example.

Since $F(s) = \int_0^\infty e^{-st} f(t)\,dt$, taking $\frac{d}{ds}$ on both sides we get:

$$\frac{d}{ds}F(s) = \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\,dt\right)$$

$$= \int_0^\infty \frac{d}{ds}\left(e^{-st} f(t)\right) dt$$

$$= \int_0^\infty -t\, e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\left(t \cdot f(t)\right) dt = -\mathcal{L}[t f(t)](s)$$

$$\Rightarrow \mathcal{L}[t f(t)](s) = (-1) \cdot \frac{d}{ds} F(s).$$

Example: Compute $\mathcal{L}[t \sin(bt)]$ and $\mathcal{L}[t^2 \sin(bt)]$.

Solution: We know

$$\mathcal{L}[\sin(bt)](s) = \frac{b}{s^2+b^2} \rightarrow F(s)$$

So using the property for $\mathcal{L}$, we have

$$\mathcal{L}[t \sin(bt)](s) = (-1)^1 \cdot \frac{d}{ds}\left(\frac{b}{s^2+b^2}\right)$$

$$= (-1) \cdot \frac{-b \cdot 2s}{(s^2+b^2)^2} = \frac{2bs}{(s^2+b^2)^2}.$$

$$\mathcal{L}[t^2 \sin(bt)](s) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{b}{s^2+b^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-2bs}{(s^2+b^2)^2}\right)$$

$$= \frac{-2b \cdot (s^2+b^2)^2 - (-2bs) \cdot 2(s^2+b^2) \cdot 2s}{(s^2+b^2)^4}$$

$$= \frac{6bs^2 - 2b^3}{(s^2+b^2)^3}.$$

Example: Compute $\mathcal{L}[te^{-2t}\cos(3t)]$.

Solution: We know

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2}, \quad s>a.$$

So $$\mathcal{L}[e^{-2t}\cos(3t)](s) = \frac{s-(-2)}{(s-(-2))^2+3^2}$$

$$= \frac{s+2}{(s+2)^2+9}, \quad s>-2.$$

Use the property for $\mathcal{L}[tf(t)](s)$:

$$\mathcal{L}[te^{-2t}\cos(3t)](s)$$

$$= (-1)^1 \cdot \frac{d}{ds}\left(\frac{s+2}{(s+2)^2+9}\right)$$

$$= (-1)\cdot\left(\frac{1\cdot((s+2)^2+9) - (s+2)\cdot 2(s+2)}{((s+2)^2+9)^2}\right)$$

$$= \frac{(s+2)^2-9}{((s+2)^2+9)^2}.$$

7.4 Inverse Laplace Transform

<u>Definition of Inverse Laplace Transform</u>: Given $F(s)$, find $f(t)$ such that $\mathcal{L}[f(t)](s) = F(s)$.

<u>Example</u>: Determine $\mathcal{L}^{-1}[F]$ where

(a) $F(s) = \frac{2}{s^3}$, (b) $F(s) = \frac{3}{s^2+9}$ (c) $F(s) = \frac{s-1}{s^2-2s+5}$

<u>Solution</u>: (a) From the table of Laplace transform, we know

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \longrightarrow$$

We want something like $\frac{2}{s^3}$, so choose n s.t. $n+1 = 3$. The constant 2 on the numerator shouldn't be considered at first.

So let $n = 2$, we get:

$$\mathcal{L}[t^2] = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{2}{s^3}\right] = t^2.$$

(b) Again from the table,

$$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2}$$

We want a denominator like s^2+9, so we choose $b = 3$.

$$\mathcal{L}[\sin(3t)] = \frac{3}{s^2+3^2} = \frac{3}{s^2+9} \Rightarrow \mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right] = \sin(3t).$$

(c) Rewrite $F(s)$: $F(s) = \frac{s-1}{s^2-2s+5} = \frac{s-1}{(s-1)^2+2^2}$

From the table we know

$$\mathcal{L}[e^{at}\cos(bt)] = \frac{s-a}{(s-a)^2+b^2}$$

To match the denominator $(s-1)^2+2^2$ in $F(s)$, we choose $a = 1$, $b = 2$ and

$$\mathcal{L}[e^t\cos(2t)] = \frac{s-1}{(s-1)^2+2^2} = F(s)$$

$$\Rightarrow \mathcal{L}^{-1}\Big[\underbrace{\frac{s-1}{(s-1)^2+2^2}}_{= \frac{s-1}{s^2-2s+5}}\Big] = e^t\cos(2t).$$

$\mathcal{L}^{-1}$ is also linear in the following sense:

- Linearity of the Inverse Transform:

(1) $\mathcal{L}^{-1}[F_1 + F_2] = \mathcal{L}^{-1}[F_1] + \mathcal{L}^{-1}[F_2]$

(2) $\mathcal{L}^{-1}[cF] = c\,\mathcal{L}^{-1}[F]$

Here c is a constant number

Example: Find $\mathcal{L}^{-1}[F]$ where

(a) $F(s) = \frac{2}{s}$ (b) $F(s) = \frac{1}{s^5}$

Solution: From the table we have:

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}.$$

If we ignore the constant, $F(s)$ contains $\frac{1}{s}$ and to match this we choose n s.t. $n+1 = 1$ (also ignore the constant in $\mathcal{L}[t^n]$ at first)

$n = 0$, $t^n = t^0 = 1$

$$\mathcal{L}[1] = \frac{0!}{s^{0+1}} = \frac{1}{s}$$

So, $\mathcal{L}^{-1}[\frac{2}{s}] = 2 \cdot \mathcal{L}^{-1}[\frac{1}{s}] = 2 \cdot 1 = 2$

Linearity of the inverse Laplace transform

(b) From the table we have

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}.$$

To match $\frac{1}{s^5}$ in $F(s)$ we choose n s.t. $n+1 = 5$, i.e. $n = 4$. Since $\mathcal{L}[t^4] = \frac{4!}{s^{4+1}} = \frac{24}{s^5}$,

we first rewrite $F(s)$: $F(s) = \frac{1}{s^5} = \frac{1}{24} \cdot \frac{4!}{s^5}$,

and

$$\mathcal{L}^{-1}[\frac{1}{s^5}] = \mathcal{L}^{-1}[\frac{1}{24} \cdot \frac{4!}{s^5}] = \frac{1}{24}\mathcal{L}^{-1}[\frac{4!}{s^5}] = \frac{1}{24} \cdot t^4$$

Linearity of inverse transform

(c) $F(s) = \frac{1}{s+4} - \frac{18}{s^2+9}$ (d) $\frac{4s+3}{s^2+25}$

Solutions: (c) Using the linearity, we know

$$\mathcal{L}^{-1}\left[\frac{1}{s+4} - \frac{18}{s^2+9}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+4}\right] - \mathcal{L}^{-1}\left[\frac{18}{s^2+9}\right].$$

For $\mathcal{L}^{-1}\left[\frac{1}{s+4}\right]$, we know from the table

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a}.$$

So let $a=-4$, $\mathcal{L}[e^{-4t}](s) = \frac{1}{s-(-4)} = \frac{1}{s+4}$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s+4}\right] = e^{-4t}.$$

Now consider $\mathcal{L}^{-1}\left[\frac{18}{s^2+9}\right]$. From the table,

$$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2}$$

letting $b=3$ (b/c $\frac{18}{s^2+9} = \frac{18}{s^2+3^2}$) we get

$$\mathcal{L}[\sin(3t)] = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}.$$

So $\mathcal{L}^{-1}\left[\frac{18}{s^2+9}\right] = \mathcal{L}^{-1}\left[6 \cdot \frac{3}{s^2+9}\right] = 6\,\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right]$

$$= 6\sin(3t).$$

Therefore $\mathcal{L}^{-1}\left[\frac{1}{s+4} - \frac{18}{s^2+9}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+4}\right] - \mathcal{L}^{-1}\left[\frac{18}{s^2+9}\right]$

$$= e^{-4t} - 6\sin(3t).$$

(d) From the table,

$$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2}, \quad \mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2}.$$

If $b=5$, $\frac{s}{s^2+b^2}$ $\qquad$ $\frac{b}{s^2+b^2}$

So we write

$$\frac{4s+3}{s^2+25} = 4\cdot\frac{s}{s^2+25} + \frac{3}{5}\cdot\frac{5}{s^2+25},$$

and $\mathcal{L}^{-1}\left[\frac{4s+3}{s^2+25}\right] = 4\mathcal{L}^{-1}\left[\frac{s}{s^2+25}\right] + \frac{3}{5}\mathcal{L}^{-1}\left[\frac{5}{s^2+25}\right]$

$$= 4\cdot\cos(5t) + \frac{3}{5}\sin(5t).$$

- **Method of Partial fractions**

Sometimes we need to find $\mathcal{L}^{-1}[F(s)]$ where

$F(s) = \frac{P(s)}{Q(s)}$, both $P(s)$ and $Q(s)$ are polynomials

There are three cases to consider:

(1) Nonrepeated linear factors

(2) Repeated linear factors

(3) Quadratic factors

(1) Nonrepeated linear factors

If $Q(s) = (s-r_1)(s-r_2)\cdots(s-r_n)$, where r_i's are different

Then write $\frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \frac{A_2}{s-r_2} + \cdots + \frac{A_n}{s-r_n}$

Example: Compute $\mathcal{L}^{-1}[F(s)]$, where

(a) $F(s) = \frac{2}{s^2+5}$ $\qquad$ (b) $F(s) = \frac{7s-1}{(s+1)(s+2)(s-3)}$

Solution: (a)

Write $\frac{2}{s^2+s} = \frac{2}{s(s+1)} = \frac{2\cdot((s+1)-s)}{s(s+1)} = \frac{2}{s} - \frac{2}{s+1}$

This is simple b/c the denominator is the product of only two $(s-r_i)$.

Then $\mathcal{L}^{-1}\left[\frac{2}{s^2+s}\right] = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{2}{s+1}\right]$

$= \mathcal{L}^{-1}\left[\frac{2}{s}\right] - \mathcal{L}^{-1}\left[\frac{2}{s+1}\right]$

$= 2\cdot\mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\,\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$ $\longrightarrow \frac{1}{s-(-1)}$

$= 2\cdot e^{0\cdot t} - 2\cdot e^{-1\cdot t} = 2 - 2e^{-t}$.

(b) Write

$$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3} \quad (\Delta)$$

(Here the denominator is the product of three terms, usually there is no easy way to find A, B, C on the right hand side. We have to solve eqs)

$$\left(\frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3} = \frac{A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)}{(s+1)(s+2)(s-3)}\right)$$

So multiplying by $(s+1)(s+2)(s-3)$ on both sides of (Δ):

$$7s-1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)$$

$$0\cdot s^2 + 7\cdot s + (-1)$$

$$\Rightarrow 7s-1 = (A+B+C)s^2 + (-A-2B+3C)s + (-6A-3B+2C)$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ -A-2B+3C=7 \\ -6A-3B+2C=-1 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = -3 \\ C = 1 \end{cases}$$

Therefore $\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}$,

$$\mathcal{L}^{-1}\left[\frac{7s-1}{(s+1)(s+2)(s-3)}\right] = \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{3}{s+2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-3}\right]$$

$$= 2e^{-t} - 3e^{-2t} + e^{3t}.$$

(2) Repeated linear factors

If $Q(s) = (s-r)^m$, then we can write

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r} + \frac{A_2}{(s-r)^2} + \cdots + \frac{A_m}{(s-r)^m}.$$

<u>Example</u>: Determine $\mathcal{L}^{-1}[F(s)]$, where

(a) $F(s) = \dfrac{s+1}{s^3}$ (b) $F(s) = \dfrac{s^2+9s+2}{(s-1)^2(s+3)}$.

<u>Solution</u>: (a) Sometimes it is not difficult to write into the form $\frac{A_1}{s-r} + \cdots + \frac{A_m}{(s-r)^m}$. For this one,

$$\frac{s+1}{s^3} = \frac{s}{s^3} + \frac{1}{s^3} = \frac{1}{s^2} + \frac{1}{s^3}.$$

So

$$\mathcal{L}^{-1}\left[\frac{s+1}{s^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^3}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1!}{s^{1+1}}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2!}{s^{2+1}}\right]$$

$$= t + \frac{1}{2}t^2.$$

(b) We want to write

$$\frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}.$$

Multiply both sides by $(s-1)^2(s+3)$ to get

$$1\cdot s^2 + 9s + 2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2$$

$$= A(s^2+2s-3) + B(s+3) + C(s^2-2s+1)$$

$$= (A+C)s^2 + (2A+B-2C)s + (-3A+3B+C)$$

$$\Rightarrow \begin{cases} A+C=1 \\ 2A+B-2C=9 \\ -3A+3B+C=2 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=3 \\ C=-1 \end{cases}$$

So $\mathcal{L}^{-1}\left[\frac{s^2+9s+2}{(s-1)^2(s+3)}\right] = \mathcal{L}^{-1}\left[\frac{2}{s-1}+\frac{3}{(s-1)^2}-\frac{1}{s+3}\right]$

$= 2\,\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + 3\cdot\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+3}\right]$

$= 2e^{t} + 3te^{t} - e^{-3t}$.

(3) Quadratic Factors

If $Q(s)$ has a factor like $(s-\alpha)^2+\beta^2$!,

then (for example)

$$\frac{C_1 s + D_1}{(s-\alpha)^2+\beta^2} = \frac{C_1(s-\alpha)}{(s-\alpha)^2+\beta^2} + \frac{(D_1 - C_1\alpha)}{(s-\alpha)^2+\beta^2}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{C_1 s + D_1}{(s-\alpha)^2+\beta^2}\right] = \underbrace{\mathcal{L}^{-1}\left[\frac{C_1(s-\alpha)}{(s-\alpha)^2+\beta^2}\right]}_{= C_1 e^{\alpha t}\cos(\beta t)} + \underbrace{\mathcal{L}^{-1}\left[\frac{D_1 - C_1\alpha}{(s-\alpha)^2+\beta^2}\right]}_{= \frac{(D_1 - C_1\alpha)}{\beta}\cdot e^{\alpha t}\sin(\beta t)}$$

Example: Determine $\mathcal{L}^{-1}[F(s)]$, where

(a) $F(s) = \frac{2s-3}{s^2+2s+5}$ (b) $F(s) = \frac{2s^2+10s}{(s^2-2s+5)(s+1)}$

Solution: (a) Write

$$F(s) = \frac{2s-3}{s^2+2s+5} = \frac{2s-3}{(s+1)^2+2^2}$$

$$= \frac{2(s+1)-5}{(s+1)^2+2^2} = 2\cdot\frac{s+1}{(s+1)^2+2^2} - \frac{5}{2}\cdot\frac{2}{(s+1)^2+2^2}$$

So $\mathcal{L}^{-1}\left[\frac{2s-3}{s^2+2s+5}\right] = \mathcal{L}^{-1}\left[2\cdot\frac{s+1}{(s+1)^2+2^2}\right] - \mathcal{L}^{-1}\left[\frac{5}{2}\cdot\frac{2}{(s+1)^2+2^2}\right]$

$= 2e^{-t}\cos(2t) - \frac{5}{2}e^{-t}\sin(2t)$

(b) In the denominator of $\frac{2s^2+10s}{(s^2-2s+5)(s+1)}$,

$s^2-2s+5 = \underline{(s-1)^2+2^2}$. Both $(s-1)$, 2 in the expression comes from $(s-1)^2+2^2$.

We try to write

$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{A(s-1)+2B}{(s-1)^2+2^2} + \frac{C}{s+1}$$

Multiply both sides by the denominator $(s^2-2s+5)(s+1)$:

$$2s^2+10s = (s+1)\cdot\big(\underbrace{A(s-1)+2B}_{As+(2B-A)}\big) + C\cdot(s^2-2s+5)$$

$$2s^2+10s = As^2+2Bs+(2B-A)+Cs^2-2Cs+5C$$

$$2s^2+10s+0 = \underbrace{(A+C)}_{\substack{\parallel\\2}}s^2+\underbrace{(2B-2C)}_{\substack{\parallel\\10}}s+\underbrace{(2B-A+5C)}_{\substack{\parallel\\0}}$$

$\Rightarrow$

$$\begin{cases} A+C=2 \\ 2B-2C=10 \\ 2B-A+5C=0 \end{cases} \Rightarrow \begin{cases} A=3 \\ B=4 \\ C=-1 \end{cases}$$

So: $\mathcal{L}^{-1}\left[\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{3(s-1)+2\cdot4}{(s-1)^2+2^2}\right] + \mathcal{L}^{-1}\left[\frac{-1}{s+1}\right]$

$$= 3\mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2+2^2}\right] + 4\mathcal{L}^{-1}\left[\frac{2}{(s-1)^2+2^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$= 3e^t\cos(2t)+4e^t\sin(2t)-e^{-t}.$$

7.5 Solving Initial Value Problems

Example: Solve the IVP

$$y'(t) = 3 \ , \ y(0) = 1.$$

Solution:

$$y' = 3$$

$$\mathcal{L}[y'] = \mathcal{L}[3]$$

Apply the property of Laplace transform in section 7.3

$$\mathcal{L}[y'](s) = s\,\mathcal{L}[y](s) - y(0)$$

(In table 7.2, line 4: $\mathcal{L}[f'](s) = s\,\mathcal{L}[f](s) - f(0)$)

$$\Rightarrow \quad s\,\mathcal{L}[y](s) - y(0) = \frac{3}{s}$$

This means a function $\mathcal{L}[y]$ of s, not product of $\mathcal{L}[y]$ and s.

$$s\,\mathcal{L}[y](s) - 1 = \frac{3}{s}$$

$$s\,\mathcal{L}[y](s) = \frac{3}{s} + 1$$

$$\mathcal{L}[y](s) = \frac{3}{s^2} + \frac{1}{s}$$

In textbook we have $Y(s)$, which is the same thing.

So

$$y = \mathcal{L}^{-1}\left[\frac{3}{s^2} + \frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{3}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s}\right]$$

$$= 3t + 1$$

Summary of the procedure:

(a) Take the Laplace transform of each side

(b) Apply the rules or properties of Laplace transforms, such as those for $\mathcal{L}[y'']$, $\mathcal{L}[t^2 y]$, and substitute all initial values.

(c) Do algebraic calculation to solve $\mathcal{L}[y]$ a function of s.

(d) Do the inverse transform to get the solution $y(t)$.

Sometimes we need to solve ODE! And use $\lim_{s\to\infty} \mathcal{L}[y](s) = 0$.

Example: Solve the IVP $y'' - 2y' - 3y = 0$ with $y(0) = 1$, $y'(0) = 4$.

Solution:

$$\mathcal{L}[y'' - 2y' - 3y] = \mathcal{L}[0]$$ ← Laplace transform on both sides

$$\mathcal{L}[y''] - 2\mathcal{L}[y'] - 3\mathcal{L}[y] = 0$$

$$(s^2\mathcal{L}[y] - sy(0) - y'(0)) - 2(s\mathcal{L}[y] - y(0)) - 3\mathcal{L}[y] = 0$$ ← Use property of Laplace transform to write $\mathcal{L}[y'']$, $\mathcal{L}[y']$.

$$s^2\mathcal{L}[y] - s - 4 - 2s\mathcal{L}[y] + 2 - 3\mathcal{L}[y] = 0$$

$$(s^2 - 2s - 3)\mathcal{L}[y] = s + 2$$

$$\mathcal{L}[y] = \frac{s+2}{s^2 - 2s - 3}$$ ← Solve $\mathcal{L}[y]$.

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\left[\frac{s+2}{s^2-2s-3}\right]$$

To solve the inverse transform, we use partial fractions and write:

$$\frac{s+2}{s^2-2s-3} = \frac{s+2}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

$$\Rightarrow \quad 1\cdot s + 2 = A(s+1) + B(s-3) = (A+B)s + (A - 3B) \quad (*)$$

$$\Rightarrow \begin{cases} 1 = A + B \\ 2 = A - 3B \end{cases} \Rightarrow \begin{cases} A = \frac{5}{4} \\ B = -\frac{1}{4} \end{cases}$$

(Alternative way from $(*)$ to solve A, B.

Let $s = -1 \Rightarrow 1\cdot(-1) + 2 = B(-1-3) = -4B \Rightarrow B = -\frac{1}{4}$.

Let $s = 3 \Rightarrow 1\cdot 3 + 2 = A(3+1) = 4A \Rightarrow A = \frac{5}{4}$)

So $\frac{s+2}{s^2-2s-3} = \frac{5}{4}\cdot\frac{1}{s-3} - \frac{1}{4}\cdot\frac{1}{s+1}$ and thus

$$y(t) = \mathcal{L}^{-1}\left[\frac{s+2}{s^2-2s-3}\right] = \frac{5}{4}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$\frac{1}{s+1} = \frac{1}{s-(-1)}$$

$$= \frac{5}{4}\cdot e^{3t} - \frac{1}{4}\cdot(e^{-t})$$

$$= \frac{5}{4}e^{3t} - \frac{1}{4}e^{-t}.$$

Example: Solve $y'' + 4y = 2t$ with $y(0) = 1$, $y'(0) = 0$

Solution:
$$\mathcal{L}[y'' + 4y] = \mathcal{L}[2t]$$
$$\mathcal{L}[y''] + 4\mathcal{L}[y] = \frac{2}{s^2}$$
$$\left(s^2\mathcal{L}[y] - s\,y(0) - y'(0)\right) + 4\mathcal{L}[y] = \frac{2}{s^2}$$
$$s^2\mathcal{L}[y] - s - 0 + 4\mathcal{L}[y] = \frac{2}{s^2}$$
$$\mathcal{L}[y]\cdot(s^2+4) = \frac{2}{s^2} + s$$
$$\mathcal{L}[y] = \frac{2}{s^2(s^2+4)} + \frac{s}{s^2+4}$$

So
$$y(t) = \mathcal{L}^{-1}\left[\frac{2}{s^2(s^2+4)}\right] + \underline{\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right]}$$

Not so complicated

For $\mathcal{L}^{-1}\left[\frac{2}{s^2(s^2+4)}\right]$, we need to write it with partial fractions
$$\frac{2}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + 2\cdot D}{s^2+4}$$
$$\Rightarrow \quad 2 = A\,s(s^2+4) + B(s^2+4) + (Cs + 2D)s^2$$
$$0\cdot s^3 + 0\cdot s^2 + 0\cdot s + 2 = (A+C)s^3 + (B+2D)s^2 + 4As + 4B$$
$$\Rightarrow \begin{cases} A+C = 0 \\ B+2D = 0 \\ 4A = 0 \\ 4B = 2 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = \frac{1}{2} \\ C = 0 \\ D = -\frac{1}{4} \end{cases}$$

Actually there is an easy to write the partial fractions in this example:
$$\frac{2}{s^2(s^2+4)} = \frac{1}{2}\cdot\frac{(s^2+4) - s^2}{s^2(s^2+4)} = \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{2}\cdot\frac{1}{s^2+4}.$$

With the expression using partial fractions, we have

$$\mathcal{L}^{-1}\left[\frac{2}{s^2(s^2+4)}\right] = \mathcal{L}^{-1}\left[\frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{4}\cdot\frac{2}{s^2+4}\right]$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= \frac{1}{2}\cdot t - \frac{1}{4}\sin(2t).$$

Also, $$\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right]$$

$$= \cos(2t)$$

Therefore

$$y(t) = \mathcal{L}^{-1}\left[\frac{2}{s^2(s^2+4)}\right] + \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \boxed{\frac{1}{2}t - \frac{1}{4}\sin(2t) + \cos(2t)}$$

<u>Example</u>: Solve the ODE $y'' + 2y' + 2y = 5$

<u>Solution</u>: To use Laplace transform to solve an ODE, we need to assume initial values $y(0) = C_1,\ y'(0) = C_2$

We have C_1, C_2 b/c we are solving a 2nd-order ODE

$$\mathcal{L}[y'' + 2y' + 2y] = \mathcal{L}[5]$$

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] + 2\mathcal{L}[y] = \frac{5}{s}$$

$$\left(s^2\mathcal{L}[y] - s\,y(0) - y'(0)\right) + 2\left(s\mathcal{L}[y] - y(0)\right) + 2\mathcal{L}[y] = \frac{5}{s}$$

$$\left(s^2\mathcal{L}[y] - C_1 s - C_2\right) + 2\left(s\mathcal{L}[y] - C_1\right) + 2\mathcal{L}[y] = \frac{5}{s}$$

$$\left(s^2 + 2s + 2\right)\cdot\mathcal{L}[y] = \frac{5}{s} + C_1(s+2) + C_2$$

$$\mathcal{L}[y] = \frac{5}{s(s^2+2s+2)} + C_1\cdot\frac{s+2}{s^2+2s+2} + C_2\frac{1}{s^2+2s+2}$$

So $$y(t) = \mathcal{L}^{-1}\left[\frac{5}{s(s^2+2s+2)}\right] + C_1\,\mathcal{L}^{-1}\left[\frac{s+2}{s^2+2s+2}\right] + C_2\,\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+2}\right].$$

As for $\mathcal{L}^{-1}\left[\frac{5}{s(s^2+2s+2)}\right]$, notice $s^2+2s+2 = (s+1)^2+1$ and write

$$\frac{5}{s(s^2+2s+2)} = \frac{A}{s} + \frac{B(s+1)+C}{s^2+2s+2}$$

$$5 = A(s^2+2s+2) + (B(s+1)+C)s$$

$$0\cdot s^2 + 0\cdot s + 5 = (A+B)s^2 + (2A+B+C)s + 2A$$

$$\Rightarrow \begin{cases} A+B = 0 \\ 2A+B+C = 0 \\ 2A = 5 \end{cases} \Rightarrow \begin{cases} A = \frac{5}{2} \\ B = -\frac{5}{2} \\ C = -\frac{5}{2} \end{cases}$$

So $$\frac{5}{s(s^2+2s+2)} = \frac{5}{2}\cdot\frac{1}{s} + \left(-\frac{5}{2}\right)\cdot\frac{s+1}{s^2+2s+2} + \left(-\frac{5}{2}\right)\cdot\frac{1}{s^2+2s+2}$$

and thus $$\mathcal{L}^{-1}\left[\frac{5}{s(s^2+2s+2)}\right] = \frac{5}{2}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{5}{2}\mathcal{L}^{-1}\left[\frac{s+1}{s^2+2s+2}\right] - \frac{5}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+2}\right]$$

$$= \frac{5}{2} - \frac{5}{2}e^{-t}\cos(t) - \frac{5}{2}e^{-t}\sin(t)$$

In addition,

$$\mathcal{L}^{-1}\left[\frac{s+2}{s^2+2s+2}\right] = \mathcal{L}^{-1}\left[\frac{(s+1)+1}{(s+1)^2+1}\right] = \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t}\cos(t) + e^{-t}\sin(t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+2}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}\sin(t)$$

Therefore

$$y(t) = \frac{5}{2} - \frac{5}{2}e^{-t}\cos(t) - \frac{5}{2}e^{-t}\sin(t) + C_1\left(e^{-t}\cos(t) + e^{-t}\sin(t)\right) + C_2 e^{-t}\sin(t).$$

Textbook example, calculation too complicated ...

Example: Solve the IVP $y''+4y'-5y=te^t$; $y(0)=1$, $y'(0)=0$.

Solution:

$$\mathcal{L}[y''+4y'-5y]=\mathcal{L}[te^t]$$

$$\mathcal{L}[y'']+4\mathcal{L}[y']-5\mathcal{L}[y]=\frac{1}{(s-1)^2}$$

$$\left(s^2\mathcal{L}[y]-s\,y(0)-y'(0)\right)+4\left(s\mathcal{L}[y]-y(0)\right)-5\mathcal{L}[y]=\frac{1}{(s-1)^2}$$

$$\left(s^2\mathcal{L}[y]-s\right)+4\left(s\mathcal{L}[y]-1\right)-5\mathcal{L}[y]=\frac{1}{(s-1)^2}$$

$$(s^2+4s-5)\mathcal{L}[y]=\frac{1}{(s-1)^2}+s+4$$

$$\Rightarrow \mathcal{L}[y]=\frac{1+(s+4)(s-1)^2}{(s-1)^2(s^2+4s-5)}$$

$(s^2+4s-5=(s+5)(s-1))$

$$=\frac{s^3+2s^2-7s+5}{(s+5)(s-1)^3}$$

$$\Rightarrow y(t)=\mathcal{L}^{-1}\left[\frac{s^3+2s^2-7s+5}{(s+5)(s-1)^3}\right]$$

Write:

$$\frac{s^3+2s^2-7s+5}{(s+5)(s-1)^3}=\frac{A}{s+5}+\frac{B}{s-1}+\frac{C}{(s-1)^2}+\frac{D}{(s-1)^3}$$

$$\Rightarrow s^3+2s^2-7s+5=A(s-1)^3+B(s+5)(s-1)^2+C(s+5)(s-1)+D(s+5)$$

$$1\cdot s^3+2s^2-7s+5=(A+B)s^3+(3B-3A+C)s^2+(3A-9B+4C+D)s+(-A+5B-5C+5D)$$

$$\Rightarrow \begin{cases} A+B=1 \\ 3B-3A+C=2 \\ 3A-9B+4C+D=-7 \\ -A+5B-5C+5D=5 \end{cases} \Rightarrow \begin{cases} A=\frac{35}{216} \\ B=\frac{181}{216} \\ C=-\frac{1}{36} \\ D=\frac{1}{6} \end{cases}$$

So $y(t) = \mathcal{L}^{-1}\left[\dfrac{s^3+2s^2-7s+5}{(s+5)(s-1)^3}\right]$

$$= \frac{35}{216}\mathcal{L}^{-1}\left[\frac{1}{s+5}\right] + \frac{181}{216}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{36}\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] + \frac{1}{6}\mathcal{L}^{-1}\left[\frac{1}{2}\cdot\frac{2}{(s-1)^3}\right]$$

$$= \frac{35}{216}e^{-5t} + \frac{181}{216}e^{t} - \frac{1}{36}te^{t} + \frac{1}{12}t^2e^{t}$$

Example: solve the IVP: $y'' + 2ty' - 4y = 1$, $y(0) = y'(0) = 0$.

Solution: First we comment that this example is a little bit different with the examples we did earlier in this section due to the term $2ty'$ in the ODE.

Apply the property in table 7.2 in section 7.3

$$\mathcal{L}[2ty'] = 2\mathcal{L}[ty'] = 2\cdot(-1)\frac{d}{ds}\left(\mathcal{L}[y']\right)$$

$$= 2\cdot(-1)\frac{d}{ds}\left(s\,\mathcal{L}[y] - y(0)\right)$$

$$= 2\cdot(-1)\frac{d}{ds}\left(s\,Y(s)\right)$$

let $\mathcal{L}[y](s) = Y(s)$.

$$= -2\left(Y(s) + s\,Y'(s)\right)$$

Based on this, we do Laplace transform for the original ODE

$$\mathcal{L}[y'' + 2ty' - 4y] = \mathcal{L}[1]$$

$$\mathcal{L}[y''] + \left(-2Y(s) - 2sY'(s)\right) - 4\mathcal{L}[y] = \frac{1}{s}$$

$$\left(s^2Y(s) - s\,y(0) - y'(0)\right) - 2Y(s) - 2sY'(s) - 4Y(s) = \frac{1}{s}$$

$$s^2Y(s) - 6Y(s) - 2sY'(s) = \frac{1}{s}$$

Divide by $(-2s)$ on both sides to get:

$$Y'(s) + \underbrace{\frac{-6+s^2}{-2s}}_{\frac{3}{s}-\frac{s}{2}}Y(s) = -\frac{1}{2s^2}$$

We need to solve the ODE:

$$Y'(s) + \underbrace{\left(\frac{3}{s} - \frac{s}{2}\right)}_{a(s)} Y(s) = \underbrace{\frac{-1}{2s^2}}_{f(s)}$$

This is a linear ODE.

$$A(s) = \int a(s)\, ds = 3\ln(s) - \frac{s^2}{4}$$

$$e^{A(s)} = e^{3\ln(s) - \frac{s^2}{4}} = e^{3\ln(s)} \cdot e^{-\frac{s^2}{4}}$$

$$= s^3 e^{-\frac{s^2}{4}}$$

By the formula for the linear ODE,

$$Y(s) = e^{-A(s)} \left(\int e^{A(s)} f(s)\, ds \right)$$

$$= \frac{1}{s^3 e^{-\frac{s^2}{4}}} \cdot \left(\int s^3 e^{-\frac{s^2}{4}} \cdot \left(\frac{-1}{2s^2}\right) ds \right)$$

$$= \frac{e^{\frac{s^2}{4}}}{s^3} \left(\int \left(-\frac{1}{2}\right) \cdot s e^{-\frac{s^2}{4}}\, ds \right)$$

$$= \frac{e^{\frac{s^2}{4}}}{s^3} \cdot \left(\int e^u\, du \right) \longleftarrow \boxed{\begin{array}{l}\text{Substitution} \\ u = -\frac{s^2}{4} \\ du = -\frac{s}{2} ds\end{array}}$$

$$= \frac{e^{\frac{s^2}{4}}}{s^3} \cdot \left(e^u + C \right)$$

$$= \frac{e^{\frac{s^2}{4}}}{s^3} \cdot \left(e^{-\frac{s^2}{4}} + C \right) = \frac{1}{s^3} + C\frac{e^{\frac{s^2}{4}}}{s^3}.$$

To find C, we need to use the property of Laplace transform that

$$\boxed{\lim_{s\to\infty}} \mathcal{L}[y](s) = \lim_{s\to\infty} \int_0^\infty e^{-st} y(t)\, dt = \boxed{0}$$

Since $\lim_{s\to\infty} Y(s) = \lim_{s\to\infty} \left(\frac{1}{s^3} + C\frac{e^{\frac{s^2}{4}}}{s^3} \right) = 0$, we see

$C = 0$. And thus: $Y(s) = \frac{1}{s^3}$.

Use inverse transform to get

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{2} \cdot \frac{2}{s^3}\right]$$

$$= \frac{1}{2} \cdot t^2 .$$

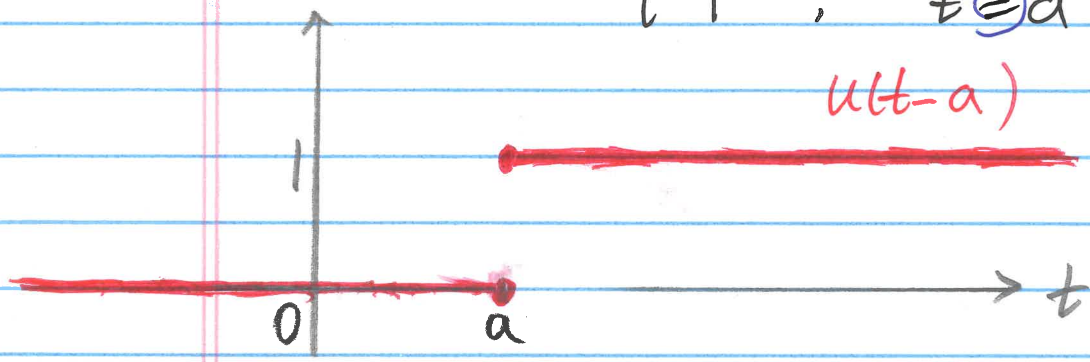
7.6 Transforms of Discontinuous Functions

Step functions: $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

For any $a \in \mathbb{R}$,

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

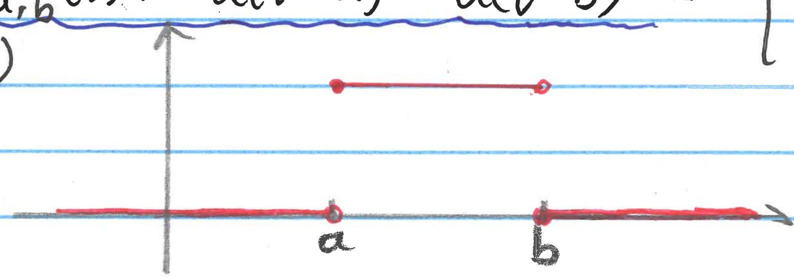
Here $t \geq a$ or $t > a$ will be treated similarly b/c in the end we are only interested in the integral of functions related.



Rectangular window function:

$$\Pi_{a,b}(t) := u(t-a) - u(t-b) = \begin{cases} 0, & t < a \\ 1, & a \leq t < b \\ 0, & t \geq b \end{cases}$$

$(a < b)$



Why we want to study $u(t-a)$, $\Pi_{a,b}(t)$? Because we can express any piecewise continuous function in term of window and step functions.

Example: Write the following functions in terms of window and step functions.

(a) $f(t) = \begin{cases} t, & 0<t<2 \\ 2, & t>2 \end{cases}$

You may wonder what $f(t)$ is when $t=2$. In this section, we do not worry about $f(t)$ at finite many points.

Solution: The part of $f(t)$ when $t<2$ is given by

$$t \cdot \Pi_{0,2}(t) \quad ,$$

and the part of $f(t)$ when $t>2$ is given by

$$2 \cdot u(t-2)$$

So

$$f(t) = t\, \Pi_{0,2}(t) + 2\, u(t-2) \; .$$

Remark: Since we'll do Laplace transform of f, only $f(t)$ with $t>0$ matters. So in the textbook you may see

$$f(t) = \begin{cases} t, & t<2 \\ 2, & t>2 \; . \end{cases}$$

This means the same thing in our setting.

(b) $f(t) = \begin{cases} \sin(t), & 0<t<\pi \\ -\sin(t), & \pi<t<2\pi \\ 0, & t>2\pi \end{cases}$

Solution:

$$f(t) = \sin(t)\, \Pi_{0,\pi}(t) - \sin(t)\, \Pi_{\pi,2\pi}(t) \; .$$

(c) $f(t) = \begin{cases} 3, & 0<t<2 \\ 1, & 2<t<5 \\ t, & 5<t<8 \\ \frac{t^2}{10}, & t>8 \end{cases}$

Solution:

$$f(t) = 3\, \Pi_{0,2}(t) + \Pi_{2,5}(t) + t\, \Pi_{5,8}(t) + \frac{t^2}{10} u(t-8)$$

Property of Laplace transform: translation in t

Theorem
$$\mathcal{L}[f(t-a)\,u(t-a)](s) = e^{-as}F(s) \qquad (8)$$

where $F(s) := \mathcal{L}[f(t)](s)$. Conversely

$$\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)\,u(t-a). \qquad (9)$$

A useful consequence of (8) is

$$\mathcal{L}[g(t)\,u(t-a)](s) = e^{-as}\mathcal{L}[g(t+a)](s) \qquad (11)$$

Proof: Not required in this class. see the textbook.

Just explain how to derive (11) from (8).

Let $f(t) = g(t+a)$, then $g(t) = f(t-a)$ and thus applying (8) to get

$$\mathcal{L}[g(t)\,u(t-a)] = \mathcal{L}[f(t-a)\,u(t-a)] = e^{-as}\mathcal{L}[f(t)](s)$$
$$= e^{-as}\mathcal{L}[g(t+a)](s).$$

We could use the property above to compute some Laplace transforms easily.

Example: Determine $\mathcal{L}[f(t)](s)$ where

(a) $f(t) = t^2 u(t-1)$

Solution: Take $g(t) = t^2$, $a=1$ and apply equation (11), then

$$g(t+a) = g(t+1) = t^2 + 2t + 1.$$

Now $\mathcal{L}[g(t+a)] = \mathcal{L}[t^2+2t+1] = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$

By formula (11), $\mathcal{L}[t^2 u(t-1)](s) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$.

(b) $f(t) = u(t-3)\,(t-3)^5$

Solution: Take "$f(t)$" in equation (8) as t^5, then by formula (8):

$$\mathcal{L}[u(t-3)(t-3)^5] = \mathcal{L}[t^5] \cdot e^{-3s} = e^{-3s} \cdot \frac{5!}{s^6}.$$

(c) $f(t) = \cos(t)\,u(t-\pi)$

Solution: Let $g(t) = \cos(t)$, $a = \pi$, then

$$g(t+\pi) = \cos(t+\pi) = -\cos(t)$$

and

$$\mathcal{L}[g(t+\pi)] = \mathcal{L}[-\cos(t)] = -\frac{s}{s^2+1}$$

Therefore by formula (11) we get

$$\mathcal{L}[\cos(t)\,u(t-\pi)] = -e^{-\pi s}\frac{s}{s^2+1}.$$

Remark: If you prefer (8) over (11), then you could proceed in the following way:

$$f(t) = \cos(t)\,u(t-\pi) = \cos\big((t-\pi)+\pi\big)\,u(t-\pi)$$
$$= -\cos(t-\pi)\cdot u(t-\pi)$$

Apply formula (8):

$$\mathcal{L}[f(t)] = e^{-\pi s} \cdot \mathcal{L}[-\cos(t)] = e^{-\pi s}\frac{-s}{s^2+1}.$$

We could also apply (9) when taking inverse Laplace transform. If we want to compute $\mathcal{L}^{-1}[e^{-as}F(s)]$,

1. first compute $f(t) = \mathcal{L}^{-1}[F(s)]$
2. Use formula (9) to get $\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)\,u(t-a)$.

Example: Compute

(a) $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right]$

Solution: In formula (9), let $F(s) = \frac{1}{s^2}$, $a = 2$,

and $f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$.

Now by (9), $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] = f(t-2)\,u(t-2)$

$$= (t-2)\,u(t-2).$$

(b) $\mathcal{L}^{-1}\left[e^{3s}\cdot\frac{5!}{s^6}\right]$

Solution: Since $\mathcal{L}^{-1}\left[\frac{5!}{s^6}\right] = t^5$, therefore by formula (9)

$$\mathcal{L}^{-1}\left[e^{3s}\frac{5!}{s^6}\right] = f(t+3)\,u(t+3).$$

These things can be useful in solving IVPs.

Example: Solve $y'' - y' - 2y = f(t)$ where

$$f(t) = \begin{cases} 0, & t < 3 \\ 7, & t \geq 3 \end{cases}$$

and $y(0) = 0$, $y'(0) = -2$

Solution: First write $f(t) = 7\,u(t-3)$ and then proceed as in section 7.5:

$$y'' - y' - 2y = 7u(t-3)$$

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] = 7\,\underbrace{\mathcal{L}[u(t-3)]}$$

For $\mathcal{L}[u(t-3)] = \mathcal{L}[1\cdot u(t-3)]$, by formula (8) we know

$$\mathcal{L}[u(t-3)] = e^{-3s}\cdot\mathcal{L}[1] = e^{-3s}\cdot\frac{1}{s}$$

So we continue

$$\left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) - \left(s\mathcal{L}[y] - y(0)\right) - 2\mathcal{L}[y] = 7\frac{e^{-3s}}{s}$$

$$s^2\mathcal{L}[y] + 2 - s\mathcal{L}[y] - 2\mathcal{L}[y] = 7e^{-3s}\frac{1}{s}$$

$$\Rightarrow \quad (s^2 - s - 2)\mathcal{L}[y] = 7e^{-3s}\frac{1}{s} - 2$$

$$\mathcal{L}[y] = 7e^{-3s}\frac{1}{s(s^2-s-2)} - \frac{2}{s^2-s-2}$$

So $\quad y(t) = 7\mathcal{L}^{-1}\left[e^{-3s}\frac{1}{s(s^2-s-2)}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s^2-s-2}\right].$

To compute this, first write

$$\frac{1}{s(s^2-s-2)} = \frac{1}{s(s+1)(s-2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-2}$$

$$1 = A(s+1)(s-2) + B(s-2)s + Cs(s+1)$$

Take $s = 0 \Rightarrow A = -\frac{1}{2}$

$s = -1 \Rightarrow B = \frac{1}{3}$

$s = 2 \Rightarrow C = \frac{1}{6}$

Therefore $\quad \mathcal{L}^{-1}\left[\frac{1}{s(s^2-s-2)}\right] = -\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{6}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]$

$$= -\frac{1}{2} + \frac{1}{3}e^{-t} + \frac{1}{6}e^{2t}$$

By property of Laplace transform in formula (9)

$$\mathcal{L}^{-1}\left[e^{-3s}\frac{1}{s(s^2-s-2)}\right] = u(t-3)\cdot\left(-\frac{1}{2} + \frac{1}{3}e^{-(t-3)} + \frac{1}{6}e^{2(t-3)}\right)$$

$$\left(\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)u(t-a)\right)$$

Also, $\mathcal{L}^{-1}\left[\frac{1}{s^2-s-2}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s-2)}\right]$

Use partial fractions

$$\frac{1}{(s+1)(s-2)} = \frac{1}{3}\cdot\frac{(s+1)-(s-2)}{(s+1)(s-2)} = \frac{1}{3}\left(\frac{1}{s-2} - \frac{1}{s+1}\right)$$

to get

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-s-2}\right] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$
$$= \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}$$

Combine all things together

$$y(t) = 7u(t-3)\cdot\left(-\frac{1}{2} + \frac{1}{3}e^{-t+3} + \frac{1}{6}e^{2t-6}\right) - \frac{2}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Example: Solve $y'' + 4y = f(t)$ with $y(0) = y'(0) = 0$,

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

Solution: Write

$$f(t) = \Pi_{0,1}(t) + (-1)\Pi_{1,2}(t)$$
$$= \left(u(t-0) - u(t-1)\right) - \left(u(t-1) - u(t-2)\right)$$
$$= u(t) - 2u(t-1) + u(t-2)$$

(recall $\Pi_{a,b}(t) = u(t-a) - u(t-b)$)

Can be computed the same way as $\mathcal{L}[u(t-3)]$ in the example above

then $\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[f]$

$$\left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) + 4\mathcal{L}[y] = \mathcal{L}[u(t)] - 2\underline{\mathcal{L}[u(t-1)]} + \underline{\mathcal{L}[u(t-2)]}$$

$$s^2\mathcal{L}[y] + 4\mathcal{L}[y] = \frac{1}{s} - 2\cdot\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}$$

(For the righthand side, maybe it's also helpful to memorize
$\mathcal{L}[u(t-a)] = \frac{1}{s}\cdot e^{-as}$)

$$(s^2+4)\mathcal{L}[y] = \frac{1}{s} - 2e^{-s}\cdot\frac{1}{s} + e^{-2s}\cdot\frac{1}{s}$$

$$\Rightarrow \mathcal{L}[y] = \frac{1}{s(s^2+4)} - 2e^{-s}\frac{1}{s(s^2+4)} + e^{-2s}\cdot\frac{1}{s(s^2+4)}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+4)}\right] - 2\mathcal{L}^{-1}\left[e^{-s}\frac{1}{s(s^2+4)}\right] + \mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s(s^2+4)}\right]$$

Let us compute $\mathcal{L}^{-1}\left[\frac{1}{s(s^2+4)}\right]$ first.

Write $\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

$$\Rightarrow 0\cdot s^2 + 0\cdot s + 1 = A(s^2+4) + (Bs+C)s = (A+B)s^2 + C\cdot s + 4A$$

$$\Rightarrow \begin{cases} A+B=0 \\ C=0 \\ 4A=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \\ C=0 \end{cases}$$

So $\mathcal{L}^{-1}\left[\frac{1}{s(s^2+4)}\right] = \mathcal{L}^{-1}\left[\frac{1}{4}\cdot\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{-\frac{1}{4}\cdot s}{s^2+4}\right]$

$$= \frac{1}{4} + \left(-\frac{1}{4}\right)\cdot\cos(2t)$$

Now we can use formula (9) to get

$$\mathcal{L}^{-1}\left[e^{-s}\frac{1}{s(s^2+4)}\right] = u(t-1)\cdot\left(\frac{1}{4} - \frac{1}{4}\cos(2(t-1))\right)$$

$$\mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s(s^2+4)}\right] = u(t-2)\cdot\left(\frac{1}{4} - \frac{1}{4}\cos(2(t-2))\right)$$

Combine everything together, we obtain

$$y(t) = \frac{1}{4} - \frac{1}{4}\cos(2t) - 2u(t-1)\cdot\left(\frac{1}{4} - \frac{1}{4}\cos(2t-2)\right) + u(t-2)\cdot\left(\frac{1}{4} - \frac{1}{4}\cos(2t-4)\right)$$

Example: Solve $y' - 2y = f(t)$ where

$$f(t) = \begin{cases} 0, & t < 3 \\ t-3, & t \geq 3 \end{cases}$$

and $y(0) = 0$.

Solution: $f(t) = u(t-3) \cdot (t-3)$

So $\mathcal{L}[f(t)] = \mathcal{L}[u(t-3) \cdot (t-3)] = e^{-3s} \mathcal{L}[t]$

$$= e^{-3s} \cdot \frac{1}{s^2}$$

$$\mathcal{L}[y' - 2y] = \mathcal{L}[f(t)]$$

$$s\mathcal{L}[y] - y(0) - 2\mathcal{L}[y] = e^{-3s} \cdot \frac{1}{s^2}$$

$$(s-2)\mathcal{L}[y] = e^{-3s} \cdot \frac{1}{s^2}$$

$$\mathcal{L}[y] = e^{-3s} \cdot \frac{1}{s^2(s-2)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left[e^{-3s} \cdot \frac{1}{s^2(s-2)}\right]$$

To compute this, first write

$$\frac{1}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2}$$

$$1 = As(s-2) + B(s-2) + Cs^2 = (A+C)s^2 + (-2A+B)s - 2B$$

$$\Rightarrow \begin{cases} A+C=0 \\ -2A+B=0 \\ -2B=1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{4} \\ B = -\frac{1}{2} \\ C = \frac{1}{4} \end{cases}$$

So $\mathcal{L}^{-1}\left[\frac{1}{s^2(s-2)}\right] = -\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]$

$$= -\frac{1}{4} - \frac{1}{2}t + \frac{1}{4}e^{2t}$$

Therefore by formula (9), we have

$$y(t) = \mathcal{L}^{-1}\left[e^{-3s} \frac{1}{s^2(s-2)}\right] = \boxed{\left(-\frac{1}{4} - \frac{1}{2}(t-3) + \frac{1}{4}e^{2(t-3)}\right) \cdot u(t-3)}.$$