Ch4 Linear Second-Order Equations

- Linear ODEs (t is independent variables, y is independent variables)

First-order $\quad y' + a_0(t)\,y = f(t)$ ← We learned how to solve this

Second-order $\quad y'' + a_1(y)\,y' + a_0(t)\,y = f(t)$

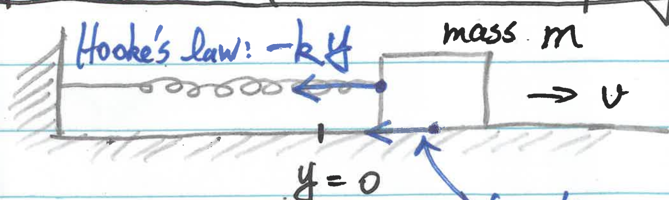
Third-order $\quad y''' + a_2(t)\,y'' + a_1(t)\,y' + a_0(t)\,y = f(t)$

$$\vdots \qquad\qquad \vdots$$

Notation: sometimes (section 5.2) we may use Dy instead of y', D^2y instead of y'' and so on.

$y' + y = f(t)$ may be written as $Dy + y = f(t)$.

4.1 Introduction: The Mass-Spring Oscillator



y: position of the object $\square$

$$\underset{\frac{d^2y}{dt^2}}{m\,a} = F = \underset{\text{force from spring}}{-ky} - \underset{\text{friction}}{bv}$$

$$\Rightarrow \quad my'' + by' + ky = 0$$

Sometimes there are other forces so the equation will become

$$my'' + by' + ky = f(t) \qquad (1)$$

Example Find the possible value of r such that $y(t) = e^{rt}$ is a solution of the ODE: $\quad y'' - y' - 2y = 0$

Solution: $\quad y(t) = e^{rt}, \quad y'(t) = re^{rt}, \quad y''(t) = r^2 e^{rt}$

So: $$y'' - y' - 2y = r^2 e^{rt} - re^{rt} - 2\cdot e^{rt}$$
$$= (r^2 - r - 2)\, e^{rt} = 0$$

$$\Rightarrow \underbrace{r^2 - r - 2}_{\substack{\parallel \\ (r-2)(r+1)}} = 0 \Rightarrow r_1 = 2\,;\ r_2 = -1.$$

or $$r_1 = \frac{1 + \sqrt{1^2 - 4\cdot 1\cdot(-2)}}{2} = 2$$

we can also use formula

$$r_2 = \frac{1 - \sqrt{1^2 - 4\cdot 1\cdot(-2)}}{2} = -1$$

We get two possible values for r: 2 and -1, and thus we see that e^{2t} and e^{-t} are the solutions (there are also other solutions).

Example $\quad y'' + y' + 25y = \sin(\Omega t)$

Find A and B such that $y(t) = A\cos(\Omega t) + B\sin(\Omega t)$ is a solution to the ODE above (Ω is a coefficient)

Solution:
$$y = A\cos(\Omega t) + B\sin(\Omega t)$$
$$y' = -A\Omega\sin(\Omega t) + B\Omega\cos(\Omega t)$$
$$y'' = -A\Omega^2\cos(\Omega t) - B\Omega^2\sin(\Omega t)$$

$$y'' + y' + 25y = \cos(\Omega t)\cdot\left(-A\Omega^2 + B\Omega + 25A\right)$$
$$+ \sin(\Omega t)\cdot\left(-B\Omega^2 - A\Omega + 25B\right) = \sin(\Omega t)$$

$$\Rightarrow \begin{cases} -A\Omega^2 + B\Omega + 25A = 1 \\ -B\Omega^2 - A\Omega + 25B = 0 \end{cases}$$

$$\begin{cases} -\Omega A + (25-\Omega^2)B = 1 \\ (25-\Omega^2)A + \Omega B = 0 \end{cases}$$ linear equations for A, B

Now we are able to solve A, B.

Second line $\Rightarrow$ $B = -\dfrac{25-\Omega^2}{\Omega} A$

Substitute into first line $\Rightarrow -\left(\Omega + \dfrac{(25-\Omega^2)^2}{\Omega}\right)A = 1$

So $A = \dfrac{-\Omega}{\Omega^2 + (\Omega^2-25)^2}$

$B = \dfrac{25-\Omega^2}{\Omega^2+(\Omega^2-25)^2}$

4.2 Homogeneous Linear Equations: The General Solution

$y'' + a_1(t)y' + a_0(t) = \boxed{0}$

- Existence theory for IVP

Recall for first-order equation:

$$\begin{cases} y' + a_0(t)y = f(t) \\ y(t_0) = y_0 \end{cases}$$

We call the ODE homogeneous if $f = 0$.

$\leftarrow$ initial value

Solution is unique if $a_0(t)$, $f(t)$ are continuous.

Second-order case:

textbook uses Y_0 and Y_1 instead of y_0, y_1.

$$\begin{cases} y'' + a_1(t)y' + a_0(t)y = f(t) \\ y(t_0) = y_0, \quad y'(t_0) = y_1 \end{cases}$$

$\leftarrow$ initial value

solution is unique if a_1, a_0, f are continuous.

Third-order linear ODE:

$$\begin{cases} y''' + a_2(t)\, y'' + a_1(t)\, y' + a_0(t)\, y = f(t) \\ \underbrace{y(t_0) = y_0,\ y'(t_0) = y_1,\ y''(t_0) = y_2}_{\text{IV}} \end{cases}$$

You see the pattern.

Examples for solving IVP

Example: We have already seen that $y'' - y' - 2y = 0$ has two solutions $Y_1(t) = e^{2t}$ and $Y_2(t) = e^{-t}$

We use capital letter Y_1, Y_2 to denote ~~solutions~~ solutions (functions) although textbook uses y_1, y_2 instead.

Try to solve the equation together with IV

$$y(0) = 3, \quad y'(0) = 0$$

Observation 1: linear combinations of $Y_1(t)$ and $Y_2(t)$ are also solutions. $\longrightarrow Y(t) = c_1 e^{2t} + c_2 e^{-t}$.

This is always true for linear ODEs with right hand side $= 0$ ($f(t)$)

(Can be checked: $Y(t) = c_1 Y_1(t) + c_2 Y_2(t)$

$$Y'' - Y' - 2Y = (c_1 Y_1 + c_2 Y_2)'' - (c_1 Y_1 + c_2 Y_2)' - 2(c_1 Y_1 + c_2 Y_2)$$

$$= c_1 (Y_1'' - Y_1' - 2Y_1) + c_2 (Y_2'' - Y_2' - 2Y_2) = c_1 \cdot 0 + c_2 \cdot 0$$

$$= 0$$)

Observation 2 : solving IVP by solving linear equations

$$Y(t) = c_1 e^{2t} + c_2 e^{-t}$$

$$Y'(t) = 2c_1 e^{2t} - c_2 e^{-t}$$

$$\cancel{Y''(t) = 4c_1 e^{2t} + c_2 e^{-t}}$$

So :

$$Y(0) = c_1 + c_2 = 3$$

$$Y'(0) = 2c_1 - c_2 = 0$$

Solving $\begin{cases} c_1 + c_2 = 3 \\ 2c_1 - c_2 = 0 \end{cases}$ gives us $\begin{cases} c_1 = 1 \\ c_2 = 2 \end{cases}$

$\Rightarrow$ solution of IVP is $e^{2t} + 2e^{-t}$.

Observation 3 : When can we do the trick above ?

For general IV : $y(0) = y_0$, $y'(0) = y_1$.

We need to solve $\begin{cases} c_1 + c_2 = y_0 \\ 2c_1 - c_2 = y_1 \end{cases}$

There is always solution if

$$\underbrace{\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}}_{\text{determinant}} = 2\cdot(-1) - 2\cdot 1 \neq 0$$

determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

So we can solve for any y_0, y_1.

But if we are given two solutions :

$$Y_1(t) = e^{2t}, \quad Y_2(t) = 3e^{2t}$$

Then consider $Y(t) = C_1 e^{2t} + C_2 \cdot 3e^{2t}$

$Y'(t) = 2C_1 e^{2t} + 2C_2 \cdot 3e^{2t}$.

For IV $y(0) = y_0$ $y'(0) = y_1$, we need to solve

$$\begin{cases} C_1 + 3C_2 = y_0 \\ 2C_1 + 6C_2 = y_1 \end{cases}$$

the determinant is from the coefficients in the linear equations

Since $\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 1 \cdot 6 - 3 \cdot 2 = 0$,

we can't guarantee a solution to IVP.

~~Exam~~ Definition: For ~~if~~ $Y_1(t)$ and $Y_2(t)$ are solutions of

$$y'' + a_1(t) y' + a_0(t) y = 0,$$

if $\begin{vmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{vmatrix} \neq 0$ for some t,

then Y_1 and Y_2 are linearly independent. Otherwise, Y_1 and Y_2 are called linealy dependent (i.e when

$$\begin{vmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{vmatrix} = 0 \text{ for any } t$$

can be proved

$$\Longleftrightarrow \begin{vmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{vmatrix} = 0 \text{ for some } t \;)$$

Remark: The definition above only for $Y_1(t)$, $Y_2(t)$ are solutions of ODE. $\begin{vmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{vmatrix}$ is called the Wronskian of Y_1 and Y_2.

Example: Show that $Y_1(t) = \sin(2t)$ and $Y_2(t) = \cos(2t)$ are two linearly independent solutions of the ODE:

can also be called fundamental pair

$$y'' + 4y = 0$$

Solution: First check they are solutions:

$$Y_1(t) = \sin(2t), \quad Y_1'(t) = 2\cos(2t), \quad Y_1''(t) = -4\sin(2t)$$

so $$Y_1'' + 4Y_1 = -4\sin(2t) + 4\sin(2t) = 0$$

$$Y_2(t) = \cos(2t), \quad Y_2'(t) = -2\sin(2t), \quad Y_2''(t) = -4\cos(2t)$$

so $$Y_2'' + 4Y_2 = -4\cos(2t) + 4\cos(2t) = 0.$$

$\Longrightarrow$ $Y_1(t)$ and $Y_2(t)$ are solutions.

Check they are independent

Wronskian: $$W[Y_1, Y_2] = \begin{vmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{vmatrix}$$

$$= -2\sin^2(2t) - 2\cos^2(2t) = -2 \neq 0$$

$\Longrightarrow$ Y_1 and Y_2 ~~are~~ is a fundamental pair.

Example (continued from above): Solve the IVP

$$\begin{cases} y'' + 4y = 0 \\ y(0) = 1, \quad y'(0) = 1. \end{cases}$$

Solution: When we have a fundamental pair.
Write the general solution

$$Y(t) = C_1 Y_1(t) + C_2 Y_2(t)$$
$$= C_1 \sin(2t) + C_2 \cos(2t)$$

Now finding the equations for C_1, C_2 to make IV satisfied

$$Y'(t) = 2C_1 \cos(2t) - 2C_2 \sin(2t)$$

$$\left.\begin{aligned} Y(0) &= C_2 = 1 \\ Y'(0) &= 2C_1 = 1 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} C_1 &= \tfrac{1}{2} \\ C_2 &= 1 \end{aligned}\right.$$

So the solution to IVP is $y(t) = \frac{1}{2}\sin(2t) + \cos(2t)$.

You can also write $Y(t)$

General Procedure to Solve Homogeneous Linear Equations

1. Find a fundamental pair $Y_1(t), Y_2(t)$
2. Write the general solution
$$Y(t) = C_1 Y_1(t) + C_2 Y_2(t)$$
and set up equations to make IV satisfied
linear equations for C_1, C_2
3. Solve C_1 and C_2, then get the solution to IVP.

Next: how to do step 1.

• Characteristic equation (auxiliary equation) :

The characteristic equation (auxiliary equation) for homogeneous linear ODE $y'' + a_1 y' + a_0 y = 0$

Now we consider a_1, a_0 are constants !

is [redacted] $\underbrace{r^2 + a_1 r + a_0}_{\text{characteristic polynomial}} = 0$.

Since this is a quadratic equation, we know how to find its roots, and there are different situations regarding the roots.

• Real Simple Roots :

"Simple" means the root only appear once when we factor the characteristic polynomial.

Example : ODE $y'' - 3y' - 10y = 0$

characteristic equation : $r^2 - 3r - 10 = 0$

$(r-5)(r+2) = 0$

$\Rightarrow$ two simple real roots : $r_1 = 5$, $r_2 = -2$.

$\Rightarrow$ a fundamental pair : $\{e^{5t}, e^{-2t}\}$.

(The procedure ~~above~~ we introduce guarantee the result is a fundamental pair, so we do not need to check the Wronskian)

As you can see, a simple root $r = r_1$ corresponds to a solution $e^{r_1 t}$.

Example : ~~$y'' - 5y' + 6y = 0$~~ $y'' - 5y' + 6y = 0$

characteristic equation $r^2 - 5r + 6 = 0$

$r_1 = 3$, $r_2 = 2$

$\Rightarrow$ a fundamental pair $\{e^{3t}, e^{2t}\}$.

"multiple" means the root appears more than once when we factor the characteristic polynomial

- Real Multiple Roots :

Example : ODE $2y'' - 20y' + 50y = 0$

$$y'' - 10y' + 25y = 0$$

So the characteristic polynomial is $r^2 - 10r + 25$.
(or you can say it is $2r^2 - 20r + 50$ from the first line)

$$r^2 - 10r + 25 = (r-5)^2$$

$\Rightarrow$ root $r = 5$ with multiplicity $m = 2$.
A fundamental pair is $\{e^{5t}, te^{5t}\}$!

Example : $y'' + 2y' + y = 0$
Characteristic equation $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0$
root $r = -1$ with multiplicity $m = 2$.
A fundamental pair is $\{e^{-t}, te^{-t}\}$.

Example : $y''' + 4y'' + 4y' = 0$
Characteristic equation $r^3 + 4r^2 + 4r = 0$

$$r(r+2)^2 = 0$$

$\Rightarrow$ simple root $r_1 = 0$ (multiplicity 1), root $r_2 = -2$ (multiplicity 2)
A fundamental set : $\{e^{0 \cdot t}, e^{-2t}, te^{-2t}\}$

Not pair because there are more than 2 solutions in the set.

Remark: We won't solve ODE with order > 2, this example is just for illustration.

Root $r = r_1$ with multiplicity m corresponds to m solutions;

$$e^{r_1 t},\ te^{r_1 t},\ \cdots,\ t^{m-1}e^{r_1 t}.$$

In the last example, we get a fundamental set

$$\{e^{0\cdot t},\ e^{-2t},\ te^{-2t}\}.$$

Notice that $e^{0\cdot t}$ corresponds to a solution $Y(t) = e^{0\cdot t} = 1$.

So the general solution of the ODE: $y''' + 4y'' + 4y' = 0$ can be given by:

$$Y(t) = C_1 \cdot 1 + C_2 e^{-2t} + C_3 te^{-2t}$$
$$= C_1 + C_2 e^{-2t} + C_3 te^{-2t}.$$

- Complex Simple Roots;

Example: $y'' - 2y' + 3y = 0$

characteristic equation $r^2 - 2r + 3 = 0$

$$r = \frac{2 \pm \sqrt{2^2 - 4\cdot 1\cdot 3}}{2} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm (\sqrt{8})i}{2}$$

Notice ($i^2 = -1$)

$$= \frac{2 \pm 2\sqrt{2}\, i}{2} = 1 \pm \sqrt{2}\, i$$

$\Rightarrow$ Two complex simple roots: $r = 1 + \sqrt{2}\, i$, $r = 1 - \sqrt{2}\, i$

Complex roots always come in pairs.
If $\alpha + \beta i$ is a root, then so is $\alpha - \beta i$.

A fundamental pair $\{e^{(\alpha+\beta i)t},\ e^{(\alpha-\beta i)t}\}$?

We can simplify! And also make the solutions real (not complex)!

Euler's formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

So $e^{(\alpha+\beta i)t} = e^{\alpha t} \cdot e^{i\beta t} = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)$

$$e^{(\alpha-\beta i)t} = e^{\alpha t} \cdot e^{-i\beta t} = e^{\alpha t}\left(\cos(-\beta t) + i\sin(-\beta t)\right)$$

$$= e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right)$$

Recall that the linear combinations of solutions are also solutions;

$$\frac{1}{2}\left(e^{(\alpha+\beta i)t} + e^{(\alpha-\beta i)t}\right) = e^{\alpha t}\cos(\beta t)$$

$$\frac{1}{2i}\left(e^{(\alpha+\beta i)t} - e^{(\alpha-\beta i)t}\right) = e^{\alpha t}\sin(\beta t)$$

$\Rightarrow$ A fundamental pair $\{e^{\alpha t}\cos(\beta t),\ e^{\alpha t}\sin(\beta t)\}$.

In the example above, $\alpha = 1$, $\beta = \sqrt{2}$, so the fundamental pair is: $\{e^{t}\cos(\sqrt{2}t),\ e^{t}\sin(\sqrt{2}t)\}$.

Example: $y'' + 4y = 0$

Characteristic equation $r^2 + 4 = 0$

Complex roots: $r = 2i$, $r = -2i$ $(\alpha = 0, \beta = 2)$

So a fundamental pair is $\{e^{0\cdot t}\cos(2t),\ e^{0\cdot t}\sin(2t)\}$

$$\parallel$$

$\{\cos(2t),\ \sin(2t)\}$.

Example : $y'' - 4y' + 5y = 0$

Characteristic equations $r^2 - 4r + 5 = 0$

$$r = \frac{4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 5}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2}$$

$$= 2 \pm i \qquad (\alpha = 2, \beta = 1)$$

A fundamental pair is $\{ e^{2t} \cos(t), e^{2t} \sin(t) \}$.

Not required

- **Complex Multiple Roots** :

Example : $y'''' + 4y''' + 14y'' + 20y' + 25y = 0$

Characteristic polynomial $r^4 + 4r^3 + 14r^2 + 20r + 25 = 0$

$$(r^2 + 2r + 5)^2 = 0$$

$r^2 + 2r + 5 = 0$ has two roots $r = -1 \pm 2i$,

so $(r^2 + 2r + 5)^2 = 0$

$$(r - (-1+2i))^2 (r - (-1-2i))^2 = 0$$

Roots : $r = -1 + 2i$ (multiplicity $m = 2$)

$r = -1 - 2i$ (multiplicity $m = 2$)

Fundamental set : $\{ e^{-t} \cos(2t), te^{-t} \cos(2t), e^{-t} \sin(2t), te^{-t} \sin(2t) \}$.

In general : Pair of roots $r = \alpha \pm \beta i$ with multiplicity m corresponds to solutions

$$e^{\alpha t} \cos(\beta t), \; t e^{\alpha t} \cos(\beta t), \; \cdots, \; t^{m-1} e^{\alpha t} \cos(\beta t),$$

$$e^{\alpha t} \sin(\beta t), \; t e^{\alpha t} \sin(\beta t), \; \cdots, \; t^{m-1} e^{\alpha t} \sin(\beta t).$$

How to find the fundamental pair for 2nd-order ODE with constant coefficient

Summary: (i) Two simple real roots: $r = r_1$, $r = r_2$,

fundamental pair $\{e^{r_1 t}, e^{r_2 t}\}$

(ii) A multiple real root $r = r_1$ with multiplicity $m = 2$.

fundamental pair $\{e^{r_1 t}, t e^{r_1 t}\}$

(iii) Complex root pair $r = \alpha \pm \beta i$,

fundamental pair $\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$.

To review how to solve IVP

Example: Solve IVP:
$$\begin{cases} y'' + 2y' + y = 0 \\ y(0) = 1, \quad y'(0) = -2 \end{cases}$$

1. Find a fundamental pair:
$$r^2 + 2r + 1 = 0$$
$$(r+1)^2 = 0$$

$\Rightarrow$ root $r = -1$ with multiplicity $m = 2$.

Fundamental pair $\{e^{-t}, te^{-t}\}$.

2. Write general solution $Y(t) = C_1 e^{-t} + C_2 t e^{-t}$.

3. Solve C_1 and C_2 based on IVs.

$$Y(t) = C_1 e^{-t} + C_2 t e^{-t}$$
$$Y'(t) = -C_1 e^{-t} + C_2 e^{-t} - C_2 t e^{-t}$$

$$\Rightarrow \begin{cases} Y(0) = C_1 + 0 = 1 \\ Y'(0) = -C_1 + C_2 - 0 = -1 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = -1 \end{cases}$$

So the solution to the IVP is:

$$\boxed{Y(t) = e^{-t} - te^{-t}}$$

4.4 Nonhomogeneous Equations : the Method of Undetermined Coefficients

This section will focus on nonhomogeneous equation

$$f(t) \neq 0$$

$$y'' + a_1(t)\, y' + a_0(t)\, y = f(t)$$

Examples :

$$y'' + 2y' - 3y = e^t$$
$$y'' + 4y = t\cos(2t)$$

• Idea of solving nonhomogeneous equation :

Consider : $y'' + 2y' - 3y = f(t)$

Suppose we know a particular solution $Y_p(t)$ of the equation above. Now we want to find all solutions, suppose $Y(t)$ is also a solution, then we plug in $Y(t) - Y_p(t)$

$$\big(Y(t) - Y_p(t)\big)'' + 2\big(Y(t) - Y_p(t)\big)' - 3\big(Y(t) - Y_p(t)\big)$$
$$= \left[Y''(t) + 2Y'(t) - 3Y(t)\right] - \left[Y_p''(t) + 2Y_p'(t) - 3Y_p(t)\right]$$
$$= f(t) - f(t) = 0$$

$\Rightarrow$ $Y(t) - Y_p(t)$ is a solution to the homogeneous version of the equation : $y'' + 2y' - 3y = 0$.

We know all solutions of the homogeneous equation are:

$$c_1 e^{-3t} + c_2 e^{t}$$

(because characteristic equation is $r^2 + 2r - 3 = 0$

$\Rightarrow \quad (r+3)(r-1) = 0 \quad \Rightarrow \quad r = \underline{1, -3}$

simple real roots

Fundamental pair $\{e^{-3t}, e^{t}\}$)

Therefore $\quad Y(t) - Y_p(t) = c_1 e^{-3t} + c_2 e^{t}$

$\Rightarrow \quad Y(t) = Y_p(t) + c_1 e^{-3t} + c_2 e^{t}$

- General Method: $\quad y'' + a_1 y' + a_0 = f(t)$

(1) Find the fundamental pair $\{Y_1, Y_2\}$ for the homogeneous version $y'' + a_1 y' + a_0 = 0$.

(2) Find one single solution Y_p for the original ODE.

(3) Then the general solutions of the original ODE is

$$Y(t) = Y_p(t) + C_1 Y_1(t) + C_2 Y_2(t)$$

Warning: the fundamental pair is for the homogeneous version, the nonhomogeneous ODE doesn't have a fundamental pair.

Examples: $\quad y'' + 4y' = 4t$

1. Homogeneous version $\quad y'' + 4y' = 0$

characteristic equation $\quad r^2 + 4r = 0$

$\quad r(r+4) = 0$

Two roots: $\quad r = 0, -4.$

$\Rightarrow$ Fundamental pair $\{\underbrace{e^{0\cdot t}}_{=1}, e^{-4t}\}$

2. Find a particular solution of the nonhomogeneous ODE
Original ODE

Consider $y_p(t) = \frac{1}{2}t^2 - \frac{1}{4}t$

$y_p'(t) = t - \frac{1}{4}$, $y_p''(t) = 1$

So $y_p'' + 4y_p' = 1 + 4(t - \frac{1}{4}) = 4t$

We'll discuss later how to find $y_p(t)$.

3. Thus the solutions to $y'' + 4y' = 4t$ are

$$y(t) = \left(\frac{1}{2}t^2 - \frac{1}{4}t\right) + c_1 + c_2 e^{-4t}.$$

Example : $y'' + y = \cos(2t)$

1. Homogeneous version $y'' + y = 0$

$r^2 + 1 = 0$

$\Rightarrow$ complex roots $r = \pm i$

$\Rightarrow$ Fundamental pair $\{\cos(t), \sin(t)\}$.

2. Find a particular solution of the original ODE

Try $y_p(t) = A\cos(2t) + B\sin(2t)$.

$y_p'(t) = -2A\sin(2t) + 2B\cos(2t)$

$y_p''(t) = -4A\cos(2t) - 4B\sin(2t)$

$$y_p'' + y_p = \left[-4A\cos(2t) - 4B\sin(2t)\right] + \left[A\cos(2t) + B\sin(2t)\right]$$

$$= (A-4A)\cos(2t) + (B-4B)\sin(2t)$$
$$= -3A\cos(2t) - 3B\sin(2t)$$
$$= \cos(2t)$$

$$\Rightarrow \begin{cases} -3A = 1 \\ -3B = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{3} \\ B = 0 \end{cases}$$

$$Y_p(t) = -\frac{1}{3}\cos(2t)$$

3. Thus the general solution of the original ODE is

$$Y(t) = -\frac{1}{3}\cos(2t) + C_1\cos(t) + C_2\sin(t).$$

- How to find the particular solution of the homogeneous equation

Example: $y'' - y = 2e^{5t}$

If $f(t) = C \cdot e^{5t}$, then $Y_p(t)$ probably looks like Ae^{5t}. Plug in:

$$(Ae^{5t})'' - Ae^{5t}$$
$$= 25Ae^{5t} - Ae^{5t} = 24Ae^{5t}$$

Set $24Ae^{5t} = 2e^{5t}$ gives us

$$24A = 2 \Rightarrow A = \frac{1}{12}.$$

So one particular solution $Y_p(t) = \frac{1}{12}e^{5t}$.

However, for $f(t) = e^{5t}$, we cannot always try to look for $Y_p(t) = Ae^{5t}$ because 5 may be a root for the characteristic equation.

If $r = 5$ is a root, for example characteristic equation is $r(r-5) = r^2 - 5r = 0$.

Nonhomogeneous equation $y'' - 5y' = e^{5t}$.

Then $(e^{5t})'' - 5(e^{5t})'$

$$= 25e^{5t} - 5 \cdot 5e^{5t} = 0$$

Of course we will get 0 b/c e^{5t} is a solution to the homogeneous equation.

General Method: 1. $y'' + a_1 y' + a_0 y = \underbrace{(c_n t^n + \cdots + c_0)}_{\text{polynomial of } t} e^{r_1 t}$

Try the form

$$Y_p(t) = t^m (A_n t^n + \cdots + A_0) e^{r_1 t}$$

where $r = r_1$ is a root of multiplicity m of the characteristic equation. Plug in Y_p to the original equation to solve $A_n, \ldots, A_0$.

$m = 0$: if $r = r_1$ is not a root

$m = 1$: if $r = r_1$ is a simple root

$m = 2$: if $r = r_1$ is a multiple root with multiplicity 2.

2\. If $f(t) = \begin{cases} (C_n t^n + \cdots + C_0)\, e^{\alpha t} \cos(\beta t) \\ \text{or } (C_n t^n + \cdots + C_0)\, e^{\alpha t} \sin(\beta t) \\ \text{or combination of the above} \end{cases}$

Use the form

$$Y_p(t) = t^m (A_n t^n + \cdots + A_0)\, e^{\alpha t} \cos(\beta t) + t^m (B_n t^n + \cdots + B_0)\, e^{\alpha t} \sin(\beta t)$$

where $m = 0$ if $\alpha \pm \beta i$ is not root of characteristic equation.
$m = 1$ if $\alpha \pm \beta i$ is a simple root.

Remark: The method above can only be applied for $f(t)$ of one of the forms above. We do not know how to find a particular solution for general $f(t)$.

Examples: $y'' + 3y' + 2y = 2e^{3t}$

1\. Homogeneous equation $y'' + 3y' + 2y = 0$

Characteristic equation: $r^2 + 3r + 2 = 0$

$(r+1)(r+2) = 0$

$\Rightarrow$ two simple roots $r = -1, -2$

Fundamental pair $\{e^{-t}, e^{-2t}\}$

2\. $f(t) = 2e^{3t}$, and $r = 3$ is not a root.

Use the form $Y_p(t) = Ae^{3t}$

$$Y_p'(t) = 3Ae^{3t}, \quad Y_p''(t) = 9Ae^{3t}$$

$$Y_p'' + 3Y_p' + 2Y_p = 9Ae^{3t} + 3 \cdot 3Ae^{3t} + 2Ae^{3t} = 20Ae^{3t}$$

Let $20Ae^{3t} = 2e^{3t} \Rightarrow A = \frac{1}{10}$.

So $Y_p(t) = \frac{1}{10}e^{3t}$.

3. General solution is

$$Y(t) = \frac{1}{10}e^{3t} + C_1e^{-t} + C_2e^{-2t}.$$

Example: $y'' - 3y' + 2y = 7e^{2t}$

1. Homogeneous equation $y'' - 3y' + 2y = 0$

$$r^2 - 3r + 2 = 0$$

$$(r-1)(r-2) = 0$$

Two simple roots $r = 1, 2$. Fundamental pair $\{e^t, e^{2t}\}$.

2. $f(t) = 7e^{2t}$, $r = 2$ is a root with multiplicity 1.

Use the form $Y_p(t) = t^1 \cdot Ae^{2t} = Ate^{2t}$

$\cancel{Y_p'' - 3Y_p' + 2Y_p} =$ $Y_p'(t) = Ae^{2t} + 2Ate^{2t}$

$$Y_p''(t) = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t}$$

$$= 4Ae^{2t} + 4Ate^{2t}$$

$$Y_p'' - 3Y_p' + 2Y_p = 4Ae^{2t} + 4Ate^{2t} - 3(Ae^{2t} + 2Ate^{2t}) + 2Ate^{2t}$$

$$= (4A - 3A)e^{2t} + (4At - 6At + 2At)e^{2t}$$

$$= Ae^{2t} = 7e^{2t}$$

$\Rightarrow A = 7$. Then $Y_p(t) = 7te^{2t}$.

3. General solution is

$$Y(t) = 7te^{2t} + C_1e^t + C_2e^{2t}.$$

Example : $y'' + 2y' + 2y = 17\cos(3t)$

1. Homogeneous equation : $y'' + 2y' + 2y = 0$

$$r^2 + 2r + 2 = 0$$

$\Rightarrow$ complex roots $r = -1 \pm 1 \cdot i$.

Fundamental pair $\{ e^{-t}\cos(t),\ e^{-t}\sin(t) \}$.

2. $r = 0 \pm 3i$ is not a root.

Use the form $Y_p(t) = A\cos(3t) + B\sin(3t)$.

Then $Y_p' = -3A\sin(3t) + 3B\cos(3t)$.

$Y_p'' = -9A\cos(3t) - 9B\sin(3t)$.

Plug into ODE :

$$-9A\cos(3t) - 9B\sin(3t) + 2(-3A\sin(3t) + 3B\cos(3t)) + 2(A\cos(3t) + B\sin(3t)) = 17\cos(3t)$$

$$\Rightarrow (-7A + 6B)\cos(3t) + (-6A - 7B)\sin(3t) = 17\cos(3t)$$

$$\Rightarrow \begin{cases} -7A + 6B = 17 \\ -6A - 7B = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{7}{5} \\ B = \frac{6}{5} \end{cases}.$$

$$Y_p(t) = -\frac{7}{5}\cos(3t) + \frac{6}{5}\sin(3t)$$

3. General solution

$$Y(t) = -\frac{7}{5}\cos(3t) + \frac{6}{5}\sin(3t) + c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$$

Example $\quad y'' - 2y' + 5y = e^t \cos(2t)$

Solution: 1. Characteristic equation of homogeneous version ODE

$$r^2 - 2r + 5 = 0$$

$$\Rightarrow \quad r = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$\alpha = 1, \beta = 2$, fundamental pair of solutions: $\{e^t \cos(2t), e^t \sin(2t)\}$

2. $r = 1 \pm 2i$ is a simple root.

Use the form $\quad Y_p(t) = t\,A e^t \cos(2t) + t \cdot B e^t \sin(2t)$

Then
$$Y_p'(t) = A e^t \cos(2t) + At\left(e^t \cos(2t) - 2e^t \sin(2t)\right) + B e^t \sin(2t) + Bt\left(e^t \sin(2t) + 2e^t \cos(2t)\right)$$

$$\begin{aligned} Y_p''(t) = {} & 2\left(A e^t \cos(2t) - A\,2e^t \sin(2t)\right) \\ & + At\left(e^t \cos(2t) - 4e^t \sin(2t) - 4e^t \cos(2t)\right) \\ & + 2\left(B e^t \sin(2t) + B\,2e^t \cos(2t)\right) \\ & + Bt\left(e^t \sin(2t) + 4e^t \cos(2t) - 4e^t \sin(2t)\right) \end{aligned}$$

So
$$\begin{aligned} Y_p'' - 2Y_p' + 5Y_p = {} & At\left(-3e^t \cos(2t) - 4e^t \sin(2t) - 2e^t \cos(2t) + 4e^t \sin(2t) + 5e^t \cos(2t)\right) \\ & + Bt\left(-3e^t \sin(2t) + 4e^t \cos(2t) - 2e^t \sin(2t) - 4e^t \cos(2t) + 5e^t \sin(2t)\right) \\ & + (2A + 4B - 2A)\, e^t \cos(2t) + (-4A + 2B - 2B)\, e^t \sin(2t) \\ = {} & 4B e^t \cos(2t) - 4A e^t \sin(2t) = e^t \cos(2t). \end{aligned}$$

$$\Rightarrow \quad B = \frac{1}{4}, \quad A = 0.$$

Therefore $Y_p(t) = \frac{1}{4} t e^{t} \sin(2t)$.

3. General solution ⇒

$$Y(t) = \frac{t}{4} e^{t} \sin(2t) + C_1 e^{t} \cos(2t) + C_2 e^{t} \sin(2t) .$$

Example : $y'' - 3y' + 2y = 5te^{4t}$, $y(0)=0$, $y'(0)=0$.

1. Characteristic equation : $r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$

Two simple roots $r = 1, 2$

Fundamental pair $\{e^{t}, e^{2t}\}$

2. $r = 4$ ⇒ not a root .

Use the form $Y_p(t) = (A_1 t + A_0) e^{4t}$.

$$Y_p' = A_1 e^{4t} + (A_1 t + A_0) 4 e^{4t}$$

$$Y_p'' = 4A_1 e^{4t} + A_1 4 e^{4t} + (A_1 t + A_0) 16 e^{4t}$$

$$= 8A_1 e^{4t} + (16At + 16B) e^{4t}$$

So
$$Y_p'' - 3Y_p' + 2Y_p = 8A_1 e^{4t} + (16At + 16B) e^{4t} - 3A_1 e^{4t} - (12A_1 t + 12A_0) e^{4t} + (2A_1 t + 2A_0) e^{4t}$$

$$= (6A_1 t + 5A_1 + 6A_0) e^{4t} = 5te^{4t}$$

$$\Rightarrow \begin{cases} 6A_1 = 5 \\ 5A_1 + 6A_0 = 0 \end{cases} \Rightarrow \begin{cases} A_1 = \frac{5}{6} \\ A_0 = -\frac{25}{36} \end{cases}$$

So $Y_p(t) = \left(\frac{5}{6}t - \frac{25}{36}\right)e^{4t}$.

3. General solution is

$$Y(t) = \left(\frac{5}{6}t - \frac{25}{36}\right)e^{4t} + c_1 e^{2t} + c_2 e^{t}.$$

To find c_1, c_2 satisfying IVs.

$$Y'(t) = \frac{5}{6}e^{4t} + \left(\frac{5}{6}t - \frac{25}{36}\right)4e^{4t} + 2c_1 e^{2t} + c_2 e^{t}$$

$$Y(0) = -\frac{25}{36} + c_1 + c_2 = 0$$

$$Y'(0) = \frac{5}{6} - \frac{25}{36}\cdot 4 + 2c_1 + c_2 = 0$$

$$\Rightarrow \begin{cases} c_1 + c_2 = \frac{25}{36} \\ 2c_1 + c_2 = \frac{35}{18} \end{cases} \Rightarrow \begin{cases} c_1 = \frac{45}{36} = \frac{5}{4} \\ c_2 = -\frac{5}{9} \end{cases}$$

So the solution of IVP is

$$\boxed{Y(t) = \left(\frac{5}{6}t - \frac{25}{36}\right)e^{4t} + \frac{5}{4}e^{2t} - \frac{5}{9}e^{t}}$$

4.5 The Superposition Principle and Undetermined Coefficients Revisited.

Theorem: If y_1 solves $y'' + a_1(t)y' + a_0(t) = f_1(t)$

and y_2 solves $y'' + a_1(t)y' + a_0(t) = f_2(t)$,

then $k_1y_1 + k_2y_2$ is a solution of

$$y'' + a_1(t)y' + a_0(t) = k_1f_1(t) + k_2f_2(t).$$

Example: $y'' + y' = t + 3e^t$, $y(0) = 0$, $y'(0) = 1$.

Solution: 1. Characteristic equation (CE) $r^2 + r = 0$

$r(r+1) = 0$

Two simple roots: $r = 0$, $r = -1$

Fundamental pair $\{e^{0\cdot t} = 1, \; e^{-t}\}$

2. $f(t) = t + 3e^t$ has two parts,

first find $Y_{p,1}(t)$ s.t. $Y''_{p,1} + Y'_{p,1} = t \Rightarrow t \cdot e^{0\cdot t}$.

Use the form $Y_{p,1}(t) = t\,(A_1 t + A_0) = A_1 t^2 + A_0 t$.

b/c $r=0$ is a simple root.

Then $Y'_{p,1} = 2A_1 t + A_0$, $Y''_{p,1} = 2A_1$,

so $Y''_{p,1} + Y'_{p,1} = 2A_1 + 2A_1 t + A_0 = 2A_1 t + (2A_1 + A_0) = t$

$$\Rightarrow \begin{cases} 2A_1 = 1 \\ 2A_1 + A_0 = 0 \end{cases} \Rightarrow \begin{cases} A_1 = \frac{1}{2} \\ A_0 = -1 \end{cases} \Rightarrow Y_{p,1}(t) = \frac{1}{2}t^2 - t$$

Next find $Y_{p,2}(t)$ s.t. $Y''_{p,2} + Y'_{p,2} = 3e^t$

Use the form $Y_{p,2}(t) = Be^t$

Then $Y'_{p,2} = Be^t$, $Y''_{p,2} = Be^t$, so

$$Y''_{p,2} + Y'_{p,2} = Be^t + Be^t = 2Be^t = 3e^t \Rightarrow B = \frac{3}{2}$$

Therefore $Y_{p,2}(t) = \frac{3}{2}e^t$

Combining $Y_{p,1}$ and $Y_{p,2}$ ~~together~~, we get

$$Y_p(t) = Y_{p,1}(t) + Y_{p,2}(t) = \frac{1}{2}t^2 - t + \frac{3}{2}e^t.$$

3. General solution is (for the ODE)

$$Y(t) = \underbrace{\tfrac{1}{2}t^2 - t + \tfrac{3}{2}e^t}_{Y_p(t)} + C_1 + C_2 e^{-t}.$$

To solve the IVP,

$$Y'(t) = t - 1 + \tfrac{3}{2}e^t + (-C_2)e^{-t}$$

So

$$Y(0) = \tfrac{3}{2} + C_1 + C_2 = 0$$

$$Y'(0) = -1 + \tfrac{3}{2} - C_2 = 1$$

$$\Rightarrow \begin{cases} C_1 = -1 \\ C_2 = -\tfrac{1}{2} \end{cases}$$

Therefore the solution to the IVP is

$$\boxed{Y(t) = \tfrac{1}{2}t^2 - t + \tfrac{3}{2}e^t - 1 - \tfrac{1}{2}e^{-t}}.$$

Example: $y'' + 2y' + 2y = 5\sin(t) + 5\cos(t) + e^{-t}$

Solution: 1. CE: $r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

Fundamental pair $\{e^{-t}\cos(t), e^{-t}\sin(t)\}$.

2. $f(t) = \underbrace{\left(5\sin(t) + 5\cos(t)\right)}_{\text{first part}} + \underbrace{e^{-t}}_{\text{second part}} = f_1(t) + f_2(t)$

First deal with $f_1(t)$, since $0 \pm 5i$ is not a solution. use the form $Y_{p,1}(t) = A\cos(t) + B\sin(t)$, then

$Y'_{p,1} = -A\sin(t) + B\cos(t)$, $Y''_{p,1} = -A\cos(t) - B\sin(t)$.

$$Y''_{p,1} + 2Y'_{p,1} + 2Y_{p,1} = -A\cos(t) - B\sin(t) + 2(-A\sin(t) + B\cos(t)) + 2(A\cos(t) + B\sin(t))$$

$$= (-A + 2B + 2A)\cos(t) + (-B - 2A + 2B)\sin(t)$$
$$= (2B + A)\cos(t) + (B - 2A)\sin(t) = 5\sin(t) + 5\cos(t)$$

$$\Rightarrow \begin{cases} 2B + A = 5 \\ B - 2A = 5 \end{cases} \Rightarrow \begin{cases} B = 3 \\ A = -1 \end{cases}$$

$$Y_{p,1}(t) = -\cos(t) + 3\sin(t)$$

Next deal with $f_2(t)$. Since -1 is not a solution, use the form $Y_{p,2}(t) = Ce^{-t}$. Then

$Y'_{p,2} = -Ce^{-t}$, $Y''_{p,2} = Ce^{-t}$.

$$Y''_{p,2} + 2Y'_{p,2} + 2Y_{p,2} = Ce^{-t} - 2Ce^{-t} + 2Ce^{-t}$$
$$= Ce^{-t} = e^{-t} \Rightarrow C = 1.$$

$$Y_{p,2}(t) = e^{-t}.$$

Therefore $Y_p(t) = Y_{p,1}(t) + Y_{p,2}(t) = -\cos(t) + 3\sin(t) + e^{-t}$.

3. General solution of the ODE is :

$$Y(t) = \underbrace{-\cos(t) + 3\sin(t) + e^{-t}}_{Y_p(t)} + C_1 e^{-t}\cos(t) + C_2 e^{-t}\sin(t)$$

4.6 Variation of Parameters

last section : $y'' + a_1 y' + a_0 y = f(t)$

$f(t) \neq 0$ but has specific forms : $(c_n t^n + \cdots + c_0)e^{rt}$

or $(c_n t^n + \cdots + c_0)e^{\alpha t}\cos(\beta t) + (D_n t^n + \cdots + D_0)e^{\alpha t}\cos(\beta t)$.

Then we can find $Y_p(t)$ and hence general solution of ODE is $Y(t) = Y_p(t) + c_1 Y_1(t) + c_2 Y_2(t)$ for any c_1, c_2.

This section, we remove restriction on $f(t)$. We can find $Y_p(t)$ for any $f(t) \neq 0$.

<u>Idea :</u> $y'' + a_1(t) y' + a_0(t) y = f(t)$

Assume we have a fundamental pair $\{Y_1, Y_2\}$ for the homogeneous version $y'' + a_1(t) y' + a_0(t) y = 0$.

To find $Y_p(t)$, use the form

$$Y_p(t) = u_1(t) Y_1(t) + u_2(t) Y_2(t).$$

Notice here u_1, u_2 are not constant numbers.

$$Y_p'(t) = [u_1'(t) Y_1(t) + u_2'(t) Y_2(t)] + [u_1(t) Y_1'(t) + u_2(t) Y_2'(t)]$$

We require $u_1'(t) Y_1(t) + u_2'(t) Y_2(t) = 0$ to avoid messy calculations.

Then $Y_p'(t) = u_1(t) Y_1'(t) + u_2(t) Y_2'(t)$

$$Y_p''(t) = u_1(t) Y_1''(t) + u_2(t) Y_2''(t) + u_1'(t) Y_1'(t) + u_2'(t) Y_2'(t)$$

$$Y_p'' + a_1 Y_p' + a_0 Y_p$$

$$= u_1(t) Y_1''(t) + u_2(t) Y_2''(t) + u_1'(t) Y_1'(t) + u_2'(t) Y_2'(t)$$

$$+ a_1(t)\big(u_1(t) Y_1'(t) + u_2(t) Y_2'(t)\big)$$

$$+ a_0(t)\big(u_1(t) Y_1(t) + u_2(t) Y_2(t)\big)$$

$$= u_1(t)\big(\underbrace{Y_1''(t) + a_1(t) Y_1'(t) + a_0(t) Y_1(t)}_{=0}\big)$$ b/c Y_1 is a sol to the homogeneous equation

$$+ u_2(t)\big(\underbrace{Y_2''(t) + a_1(t) Y_2'(t) + a_0(t) Y_2(t)}_{=0}\big)$$ b/c Y_2 is a sol to the homogeneous equation

$$+ \big(u_1'(t) Y_1'(t) + u_2'(t) Y_2'(t) \big)$$

$$= u_1'(t) Y_1'(t) + u_2'(t) Y_2'(t) = f(t)$$

Therefore in order to make $Y_p(t) = u_1(t) Y_1(t) + u_2(t) Y_2(t)$ a particular solution to the nonhomogeneous ODE, we need

$$\begin{cases} u_1' Y_1 + u_2' Y_2 = 0 \\ u_1' Y_1' + u_2' Y_2' = f(t) \end{cases} \qquad (*)$$

In matrix form, it can be written as

$$\begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

Using linear algebra:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W[Y_1, Y_2]} \begin{bmatrix} -Y_2 f(t) \\ Y_1 f(t) \end{bmatrix}$$

where $W[Y_1, Y_2] = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix} = Y_1 Y_2' - Y_2 Y_1'$.

Thus

$$u_1' = -\frac{Y_2(t) f(t)}{W[Y_1, Y_2]}, \quad u_2' = \frac{Y_1(t) f(t)}{W[Y_1, Y_2]} \qquad (**)$$

Once we solve $u_1'(t)$ and $u_2'(t)$ through either (*) or (**), we can then determine $u_1(t)$ and $u_2(t)$. In the end, substitute $u_1(t)$ and $u_2(t)$ into the expression for $Y_p(t)$ to obtain a particular solution $Y_p(t)$.

Here is a formula for $Y_p(t)$

$$(\Delta) \quad Y_p(t) = Y_1 \int \frac{-Y_2(t) f(t)}{W[Y_1, Y_2]} dt + Y_2 \int \frac{Y_1(t) f(t)}{W[Y_1, Y_2]} dt$$

Summary of Method for finding $Y_p(t)$

Use formula (Δ) or follows procedure below:

(1) Find a fundamental pair $\{Y_1(t), Y_2(t)\}$

(2) Write $Y_p(t) = u_1(t) Y_1(t) + u_2(t) Y_2(t)$

(3) Solve u_1', u_2' from (*), then solve u_1, u_2

(4) Plug in to find $Y_p(t)$.

Examples: $y'' + y = \sec(t)$ $\longrightarrow$ $\sec(t) = \frac{1}{\cos(t)}$

Solution: 1. CE: $r^2 + 1 = 0 \Rightarrow r = \pm i$

Fundamental pair $\{\cos(t), \sin(t)\}$

2. $Y_p(t) = u_1(t) Y_1(t) + u_2(t) Y_2(t)$,

$$\begin{cases} u_1' \cdot \underset{\overset{\parallel}{Y_1}}{\cos(t)} + u_2' \underset{\overset{\parallel}{Y_2}}{\sin(t)} = 0 \\ u_1' \cdot \underset{\underset{Y_1'}{\parallel}}{(-\sin(t))} + u_2' \underset{\underset{Y_2'}{\parallel}}{\cos(t)} = \sec(t) \end{cases}$$

First line: $u_2' = -\frac{\cos(t)}{\sin(t)} u_1'$

$$-u_1' \sin(t) - \frac{\cos^2(t)}{\sin(t)} u_1' = -\frac{1}{\sin(t)} u_1' = \sec(t)$$

$$\Rightarrow \quad u_1' = -\tan(t), \quad u_2' = 1.$$

substitution $v = \cos(t)$, $dv = -\sin(t)dt$

$$u_1 = \int -\tan(t)\,dt = -\int \frac{\sin(t)}{\cos(t)}\,dt = \int \frac{1}{v}\,dv$$

$$= \ln|v| = \ln|\cos(t)| \; (+ \; C)$$

$$u_2 = \int 1\,dt = t \; (+ \; C)$$

$$Y_p(t) = u_1 Y_1 + u_2 Y_2 = \ln|\cos(t)|\cos(t) + t\sin(t)$$

General solution is

$$Y(t) = \ln|\cos(t)|\cos(t) + t\sin(t) + C_1\cos(t) + C_2\sin(t).$$

<u>Examples</u>: $y'' + y = \tan(t)$

1. CE: $r^2 + 1 = 0 \Rightarrow r = \pm i$

Fundamental pair $\{\cos(t), \sin(t)\}$

2. $Y_p(t) = u_1(t)Y_1(t) + u_2(t)Y_2(t)$, where $Y_1(t) = \cos(t)$, $Y_2(t) = \sin(t)$

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

$$\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \tan(t) \end{pmatrix}$$

$$\Rightarrow u_1' = -\frac{\sin^2(t)}{\cos(t)}, \quad u_2' = \sin(t)$$

$$u_1 = \int \frac{-\sin^2(t)}{\cos(t)}\,dt = \int \frac{-\sin^2(t)}{\cos^2(t)}\cos(t)\,dt$$

substitution $v = \cancel{\cos(t)}$ $\sin(t)$

$$= -\int \frac{v^2}{1-v^2}\,dv = +\int +1 + \frac{-1}{2(1-v)} + \frac{-1}{2(1+v)}\,dv$$

$$= +v + \frac{1}{2}\ln|1-v| - \frac{1}{2}\ln|1+v|$$

$$= \sin(t) - \frac{1}{2}\ln\left|\frac{1+\sin(t)}{1-\sin(t)}\right| \cancel{+} \; C$$

can also be written as $\ln|\sec(t) + \tan(t)|$

$$u_2 = \int \sin(t)\,dt = -\cos(t) \cancel{+C}$$

So $$Y_p(t) = u_1 Y_1 + u_2 Y_2$$
$$= \left(\sin(t) - \frac{1}{2}\ln\left|\frac{1+\sin(t)}{1-\sin(t)}\right|\right)\cos(t) - \cos(t)\cdot\sin(t)$$
$$= -\frac{1}{2}\cos(t)\ln\left|\frac{1+\sin(t)}{1-\sin(t)}\right|.$$

So the general solution of ODE is

$$Y(t) = -\frac{1}{2}\cos(t)\ln\left|\frac{1+\sin(t)}{1-\sin(t)}\right| + C_1\cos(t) + C_2\sin(t)$$

Example : $(t^2+1)y'' - 2ty' + 2y = (t^2+1)^2$

Find a particular solution for the equation above, suppose we are given a fundamental pair $\{t, t^2-1\}$.

Not constant coefficients, we do not know how to find a fundamental pair.

★Write ODE as : $y'' - \frac{2t}{(t^2+1)}y' + \frac{2}{t^2+1}y = \overbrace{(t^2+1)}^{f(t)}$

Solution : Let's try to use formula ($\triangle$) directly.

$$W[Y_1, Y_2] = \begin{vmatrix} t & t^2-1 \\ 1 & 2t \end{vmatrix} = t\cdot 2t - (t^2-1) = t^2+1$$

$u_1 \rightarrow$ $$\int \frac{-Y_2(t)f(t)}{W[Y_1,Y_2]}dt = \int \frac{-(t^2-1)(t^2+1)}{t^2+1}dt = \int (1-t^2)dt = t - \frac{1}{3}t^3 \cancel{+C}$$

$u_2 \rightarrow$ $$\int \frac{Y_1(t)f(t)}{W[Y_1,Y_2]}dt = \int \frac{t(t^2+1)\,dt}{t^2+1} = \int t\,dt = \frac{1}{2}t^2 \cancel{+C}$$

$$Y_p(t) = \left(t - \frac{1}{3}t^3\right)t + \left(\frac{1}{2}t^2\right)\cdot(t^2-1) = \frac{1}{6}t^4 + \frac{1}{2}t^2.$$

We can make this example an IVP by adding IVs:

$$y(1)=0\ ,\quad y'(1)=1\ .$$

Since general solution is

$$Y(t) = Y_p(t) + c_1 t + c_2(t^2-1)$$
$$= \frac{1}{6}t^4 + \frac{1}{2}t^2 + c_1 t + c_2(t^2-1)$$

Then $Y'(t) = \frac{2}{3}t^3 + t + c_1 + 2c_2 t$

$$Y(1) = \frac{1}{6} + \frac{1}{2} + c_1 = 0$$
$$Y'(1) = \frac{2}{3} + 1 + c_1 + 2c_2 = 1$$

$\Rightarrow \quad c_1 = -\frac{2}{3}\ ,\quad c_2 = 0$

So the solution to the IVP is $\quad Y(t) = \frac{1}{6}t^4 + \frac{1}{2}t^2 - \frac{2}{3}t$.

<u>Remark</u>: The method introduced in this section is general. So it can be also applied to $f(t)$ of the form in last section. However, the method of undetermined coefficients is usually easier. Let's do an example we did in the last section;

$$y'' + 3y' + 2y = 2e^{3t}$$

<u>Solution</u>: CE: $r^2+3r+2=0 \Rightarrow r=-1,-2$

Fundamental pair $\{\underset{Y_1}{\underset{\uparrow}{e^{-t}}}, \underset{Y_2}{\underset{\uparrow}{e^{-2t}}}\}$

$$W[Y_1,Y_2] = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = e^{-t}\cdot(-2e^{-2t}) - e^{-2t}\cdot(-e^{-t})$$

$$= -2e^{-3t} + e^{-3t} = -e^{-3t}$$

$$\int \frac{-Y_2(t) f(t)}{W[Y_1, Y_2]} dt = \int \frac{-e^{-2t} \cdot 2e^{3t} dt}{-e^{-3t}} = \int 2e^{4t} dt$$

$$= \frac{1}{2} e^{4t} \cancel{+ C}$$

$$\int \frac{Y_1(t) f(t)}{W[Y_1, Y_2]} dt = \int \frac{e^{-t} \cdot 2e^{3t} dt}{-e^{-3t}} = \int -2e^{5t} dt$$

$$= -\frac{2}{5} e^{5t} \cancel{+ C}$$

So: $Y_p(t) = \frac{1}{2} e^{4t} \cdot e^{-t} + \left(-\frac{2}{5} e^{5t}\right) \cdot e^{-2t}$

$$= \frac{1}{2} e^{3t} - \frac{2}{5} e^{3t} = \frac{1}{10} e^{3t}$$

General solution to the ODE is $Y(t) = \frac{1}{10} e^{3t} + C_1 e^{-t} + C_2 e^{-2t}$ □

I think this process is more complicated than the one we did using the method of undetermined coefficients. However, if you prefer the method here, you're free to use it for any $f(t)$.

4.7 Variable-Coefficient Equations

Cauchy-Euler Equation: $a_2 t^2 y'' + a_1 t y' + a_0 y = f(t)$

or $a t^2 y'' + b t y' + c y = f(t)$.

Examples: $3t^2 y'' + 11 t y' - 3y = \sin(t)$ ✓

$t^2 y'' + 5t y' + 5y = 0$ ✓

Not Cauchy-Euler → $2y'' - 3t y' + 11 y = 3t - 1$ ✗

Next we show how to find a fundamental pair $\{Y_1, Y_2\}$ for the homogeneous equation $a t^2 y'' + b t y' + c y = 0$.