3.2 Compartmental Analysis

I. Mixing Problems

Introduction: Tank contains saltwater

Saltwater pumped in $\longrightarrow$ $\boxed{\text{saltwater}}$ $\longrightarrow$ Saltwater pumped out

Input Output

Goal: know how much salt inside tank at any time t.

Approach: Let Q be the amount of salt at time t ← Not saltwater!

$$\frac{dQ}{dt} = \text{Input Rate for salt} - \text{Output Rate for salt}$$

Sometimes hard to figure out, always functions of t, Q.

Example: A tank initially contains 500L of saltwater with a concentration of 0.2 kg/L. Saltwater with a concentration of 0.3 kg/L is being pumped in at 10 L/min, while the tank is being emptied of the mixture at the same rate. Find the amount of salt in the tank at time t.

Solution: Let $Q(t)$ be the amount of salt at time t.

How to set up an IVP for Q?

1. IV: $Q(0) = 500L \cdot 0.2\,kg/L = 100\ (kg)$

Initial Volume of Saltwater × Initial Concentration

Note: Salt = Saltwater × Concentration !

2. Find input rate and output rate for salt.

Input rate for salt = Input rate for saltwater × Concentration (input)

$$10L/min \cdot 0.3kg/L = 3 \ kg/min$$

Output rate for salt = Output rate for saltwater × Concentration (tank)

$$10L/min \cdot \frac{\text{Total Salt in tank}}{\text{Total Saltwater in tank}} = \frac{10Q}{500}$$

$$= \frac{Q}{500} \ kg/L$$

Why? Because the saltwater is always 500L since pumped in rate is the same as pumped out rate.

3. Write ODE together with IV

$$\begin{cases} \frac{dQ}{dt} = 3 - \frac{10Q}{500} = 3 - 0.02Q \\ Q(0) = 100 \end{cases}$$

Don't forget initial value.

Now we need to solve the IVP.

$$\frac{dQ}{3-0.02Q} = dt$$

$$\int \frac{dQ}{3-0.02Q} = \int dt$$

$$-\frac{1}{0.02} \ln|3-0.02Q| = t + C$$

$$\ln|3-0.02Q| = -0.02t - 0.02C$$

$$|3-0.02Q| = e^{-0.02t-0.02C}$$

$$3-0.02Q = C_1 e^{-0.02t}$$

another constant, can also denote by C

$$\Rightarrow Q(t) = 150 - 50C_1 e^{-0.02t}$$

$$= 150 - C_2 e^{-0.02t}$$

Use IV, $100 = 150 - C_2 e^{-0.02 \cdot 0} = 150 - C_2$

$$\Rightarrow C_2 = 50$$

So $Q(t) = 150 - 50e^{-0.02t}$

Example : A 300 gal tank initially contains 200 gal of saltwater with concentration 0.15 lb/gal. Saltwater with a concentration 0.2 lb/gal is being pumped in at 6 gal/min, while the tank is being emptied of the mixture at 4 gal/min. How much salt will be in the tank when it is full?

key observation : Volume of saltwater is changing !

Volume at time t is $200 + (6-4)t = 200 + 2t$ gal

Q : when will the tank be full ?

$200 + 2t = 300 \Rightarrow t = 50$.

Let Q be the amount of salt, start to set up the IVP.

1. IV : $Q(0) = \underbrace{200 \text{ gal}}_{\text{Saltwater}} \cdot \underbrace{0.15 \text{ lb/gal}}_{\text{Concentration}} = 30 \text{ lb}$

2. Input rate : $(6 \text{ gal/min})(0.2 \text{ lb/gal}) = 1.2 \text{ lb/min}$

Output rate : $(4 \text{ gal/min})\left(\frac{Q}{200+2t} \text{ lb/gal}\right) = \frac{4Q}{200+2t} \text{ lb/min}$

Concentration in tank at time t : $\frac{\text{Salt}}{\text{Saltwater}}$.

3. $$\begin{cases} \frac{dQ}{dt} = 1.2 - \frac{4Q}{200+2t} = 1.2 - \frac{2}{100+t} Q \\ Q(0) = 30 \end{cases}$$

Now start to solve the IVP.

$$\frac{dQ}{dt} + \underbrace{\left(\frac{2}{100+t}\right)}_{a(t)} Q = 1.2$$ ← standard form for 1st order linear ODE.

$$A(t) = \int a(t)\,dt = \int \frac{2}{100+t}\,dt = 2\ln(100+t)$$

$$e^{A(t)} = e^{2\ln(100+t)} = \left(e^{\ln(100+t)}\right)^2 = (100+t)^2$$

$$Q(t) = e^{-A(t)}\left(\int e^{A(t)} \cdot 1.2\,dt\right)$$

$$= \frac{1}{(100+t)^2}\left(\int 1.2(t+100)^2\,dt\right)$$

$$= \frac{1}{(100+t)^2}\left(0.4(t+100)^3 + C\right)$$

$$= 0.4(t+100) + \frac{C}{(t+100)^2}$$

Plug in IV $(t=0, Q=30)$ and get

$$30 = 0.4 \cdot 100 + \frac{C}{100^2} \Rightarrow C = -10^5$$

$$\Rightarrow \quad Q(t) = 0.4(t+100) - 10^5 (t+100)^{-2}$$

So

$$Q(50) = 0.4 \cdot 150 - 10^5 \cdot 150^{-2}$$

$$= 60 - \frac{10^5}{150^2} = \frac{500}{9}$$

II. Population Models:

Introduction: population of a certain species changes due to $\underbrace{\text{growth}}_{\text{Input Rate}}$ and $\underbrace{\text{death}}_{\text{Output Rate}}$.

Goal: know population at any time t

Approach: $\dfrac{dp}{dt} = \text{Input Rate} - \text{Output Rate}$ (p is population)

Example: A species with population p_0 initially, has a growth rate 5% and death rate 3%.

Input rate $= 5\% \cdot p$! Output rate $= 3\% \cdot p$

$$\begin{cases} \dfrac{dp}{dt} = 0.05p - 0.03p = 0.02p \\ p(0) = p_0 \end{cases}$$

Solve the IVP:

$$\int \frac{dp}{p} = \int 0.02\, dt$$

$$\ln|p| = 0.02t + C$$

$$\Rightarrow \quad p(t) = e^{0.02t} \cdot C_1$$

Plug in IV: $p_0 = e^{0.02 \cdot 0} \cdot C_1 \Rightarrow C_1 = p_0$

So $p(t) = p_0 e^{0.02t}$.

The example above is an example of Malthusian model:

$$\begin{cases} p'(t) = (k_1 - k_2)p = kp \\ p(0) = p_0 \end{cases}$$

<u>Example</u>: logistic model.

$$\begin{cases} \dfrac{dp}{dt} = k_1 p - k_3 \dfrac{p(p-1)}{2} \\ p(0) = p_0 \end{cases}$$

two-party interaction.

Solve IVP:

$$\frac{dp}{dt} = -\frac{k_3}{2} p \left(p - 1 - \frac{2k_1}{k_3} \right)$$

$$= -A p (p - p_1) \quad \text{where } A = \frac{k_3}{2}, \ p_1 = \frac{2k_1}{k_3} + 1.$$

$$\frac{dp}{p(p-p_1)} = -A\,dt$$

$$\int \frac{dp}{p(p-p_1)} = \int -A\,dt$$

$$\frac{1}{p_1} \ln \left| \frac{p-p_1}{p} \right| = -At + C$$

$$\left| \frac{p-p_1}{p} \right| = e^{-Ap_1 t} \cdot e^{p_1 C}$$

$$\frac{p-p_1}{p} = C e^{-Ap_1 t}$$

$$\Rightarrow \quad p = \frac{p_1}{1 - C e^{-Ap_1 t}}$$

Use IV we could find $C = 1 - \frac{p_1}{p_0}$.

Example: In a certain neighborhood there is a mosquito problem. The population starts at 10 M and has a growth rate of 20% monthly. Traps are put out and these traps kill 3 M monthly. Find when the mosquitos will be wiped out.

Solution: Let $p(t)$ be the number of mosquitos in million.

$$\begin{aligned} &\text{IV}: \\ &\text{ODE}: \end{aligned} \begin{cases} p(0) = 10 \\ p'(t) = \underbrace{0.2p}_{\text{Input}} - \underbrace{3}_{\text{Output}} \end{cases}$$

To solve IVP,

$$\frac{dp}{dt} + \underbrace{(-0.2)}_{a(t) = -0.2} p = \underbrace{-3}_{b(t)}$$

$$A(t) = \int a(t)\,dt = \int -0.2\,dt = -0.2t$$

$$e^{A(t)} = e^{-0.2t}$$

$$\begin{aligned} p(t) &= e^{-A(t)} \left(\int e^{A(t)} b(t)\,dt \right) \\ &= e^{0.2t} \left(\int e^{-0.2t} (-3)\,dt \right) \\ &= e^{0.2t} \left(\frac{-3}{-0.2} e^{-0.2t} + C \right) \\ &= 15 + Ce^{0.2t} \end{aligned}$$

Plug in IV $(t=0,\ p=10)$: $\quad 10 = 15 + Ce^{0.2\cdot 0} = 15 + C$

$\Rightarrow C = -5$, so $p(t) = 15 - 5e^{0.2t}$.

Q : When will mosquitos be wiped out ?

$$p(t) = 15 - 5e^{0.2t} = 0 \Rightarrow e^{0.2t} = 3 \Rightarrow t = 5\ln(3)$$

3.4 Newtonian Mechanics

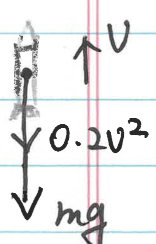
Newton's law : $m\frac{dv}{dt} = F$

acceleration

gravity, air resistence, ...
functions of t, v

Example : An object of mass 1kg is shot upward from the ground with an initial velocity 7 m/s. If the magnitude of the force due to air resistance is $0.2v^2$, when will the object reach its maximum height above the ground ?

This is the time when $v = 0$!



$$\begin{cases} m\frac{dv}{dt} = -mg - 0.2v^2 \\ v(0) = 7 \end{cases}$$

$$\frac{dv}{dt} = -g - 0.2v^2$$

$$\frac{dv}{g + 0.2v^2} = -dt$$

$$\int \frac{5\,dv}{5g + v^2} = \int -dt$$

$$\frac{5}{\sqrt{5g}} \arctan\left(\frac{v}{\sqrt{5g}}\right) = -t + C$$

Since $\sqrt{5g} = \sqrt{5 \cdot 9.8} = 7$, so : $\frac{5}{7}\arctan\left(\frac{v}{7}\right) = -t + C$

$$\arctan\left(\frac{v}{7}\right) = -\frac{7}{5}t + C_1$$

Plug in IV : $\arctan\left(\frac{7}{7}\right) = -\frac{7}{5}\cdot 0 + C_1$

$$\Rightarrow \quad C_1 = \arctan(1) = \frac{\pi}{4}$$

So : $\arctan\left(\frac{v}{7}\right) = -\frac{7}{5}t + \frac{\pi}{4}$

$$\frac{v}{7} = \tan\left(\frac{\pi}{4} - \frac{7}{5}t\right)$$

$$v = 7\tan\left(\frac{\pi}{4} - \frac{7}{5}t\right)$$

Solve $v(t) = 7\tan\left(\frac{\pi}{4} - \frac{7}{5}t\right) = 0$

$$\Rightarrow \boxed{t = \frac{5\pi}{28}}$$