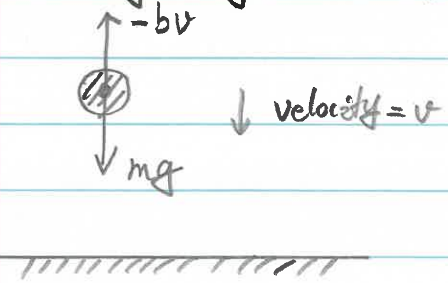
Aug 29, 2019

2.1 Introduction : Motion of a Falling Body

• Falling body with air resistance (we have seen this in section 1.1)



Instead of what we did in section 1.1 to write an ODE for height h, here we want to write an ODE for velocity v.

Newton's second law
$$m\underset{\frac{dv}{dt}}{\underset{\parallel}{a}} = \underset{mg-bv}{\underset{\parallel}{F}}$$

So:
$$m\frac{dv}{dt} = mg - bv \Rightarrow \frac{dv}{dt} = \frac{mg-bv}{m}$$

We are interested in solving the following IVP (v_0 is given)
$$\begin{cases} \dfrac{dv}{dt} = \dfrac{mg-bv}{m} \\ v(0) = v_0 \end{cases} \quad (1)$$

Not required to read. Just a justification of (1), is the ~~same as~~ equivalent to the ODE in section 1.1.

Remark: In section 1.1 of the note, we obtain an ODE
$$\frac{d^2h}{dt^2} = -g - \frac{K}{m}\frac{dh}{dt}.$$

There K is the coefficient the same as b in (1), so replacing it by b, what we got is
$$\frac{d^2h}{dt^2} = -g - \frac{b}{m}\frac{dh}{dt} \quad (2)$$

However, you may still wonder the sign on the RHS of (2), which seems to be different compared to (1). This is because
$$\frac{dh}{dt} = -v \quad (3)$$
, where the negative sign is because that we assume a velocity downward is positive in this section and eq (1).

In fact, using (3) we know $\frac{d^2h}{dt^2} = -\frac{dv}{dt}$ (4), and plugging (3) and (4) into (2) we get:

$$-\frac{dv}{dt} = -g - \frac{b}{m}(-v)$$

$$\Longrightarrow \frac{dv}{dt} = g + \frac{b}{m}(-v) = \frac{mg - bv}{m}$$

which is the same ODE in (1).

2.2 Separable Equation

We will learn how to solve a certain type of ODE in this section, and in particular we can use this technique to solve eq (1)

- Definition of Separable ODE:

  An ODE is separable if it can be written in the form

  $$\frac{dy}{dx} = g(x)\,p(y) \quad . \quad (2)$$

~~Exam~~ The word separable comes from the fact that the right side is separated into a product of a function of x and a function of y.

- ~~Solving for Non-Constant~~ Examples:

(1) $\frac{dy}{dx} = xy$ separable! Already separated

(2) $xy' + y' = y^2$ separable b/c we can rewrite it as:

$$(x+1)y' = y^2 \Rightarrow y' = \frac{y^2}{x+1} = \left(\frac{1}{x+1}\right)\cdot y^2 .$$

(3) $y' = t + y$ (notice t is independent variable here)

Not separable b/c we cannot write the RHS as a product we want

(4) $\frac{dy}{dx} = 1 + xy$. Not separable b/c cannot write as the form in definition.

• Methods: Solving for Non-Constant Solutions

We first look at one example before discussing the general procedure

Example: solve the following IVP

$$\frac{dy}{dx} = \frac{x-5}{y^2}, \quad y(0) = 2. \quad (3)$$

Solution: Recall that in order to solve IVP we need to find the general solution of the ODE first.

$$\frac{dy}{dx} = \frac{x-5}{y^2}$$

$$y^2\,dy = (x-5)\,dx$$

$$\int y^2\,dy = \int (x-5)\,dx$$

$$\frac{1}{3}y^3 + C_1 = \frac{1}{2}(x-5)^2 + C_2$$

$$\frac{1}{3}y^3 = \frac{1}{2}(x-5)^2 + C$$

why? since C_1, C_2 can be any constants, so $C_2 - C_1$ is also a constant can be chosen, therefore we only have one constant

$$\Rightarrow \quad y = \left(\frac{3}{2}(x-5)^2 + 3C\right)^{\frac{1}{3}}$$

You may also write the solution as $y = \left(\frac{3}{2}(x-5)^2 + \tilde{C}\right)^{\frac{1}{3}}$ by replacing $3C$ with $\tilde{C}$, or $y = \left(\frac{3}{2}x^2 - 15x + \tilde{C}\right)^{\frac{1}{3}}$

b/c $\frac{3}{2}(x-5)^2 + 3C = \frac{3}{2}(x^2 - 10x + 25) + 3C$

$$= \frac{3}{2}x^2 - 15x + \underbrace{\frac{75}{2} + 3C}_{\text{this can be any number, so we use } \tilde{C}}$$

Say, you like writing the general solution as

$$y = \left(\frac{3}{2}(x-5)^2 + \tilde{c}\right)^{\frac{1}{3}} \quad (4)$$

Once you have found the general solution, you still need to find the constant in order to solve the IVP.

Since IV is $y(0) = 2$, we plug $x=0$, $y=2$ into (4) and get

$$2 = \left(\frac{3}{2}(-5)^2 + \tilde{c}\right)^{\frac{1}{3}}$$

$$8 = \frac{3}{2} \cdot 25 + \tilde{c}$$

$$\Rightarrow \quad \tilde{c} = 8 - \frac{3}{2} \cdot 25 = -\frac{59}{2}$$

So the particular solution of IVP is

$$y = \left(\frac{3}{2}(x-5)^2 - \frac{59}{2}\right)^{\frac{1}{3}}.$$

(If you want to expand $(x-5)^2 = x^2 - 10x + 25$, you will get $y = \left(\frac{3}{2}x^2 - 15x + 8\right)^{\frac{1}{3}}$)

General Procedure:

$$\frac{dy}{dx} = g(x)\, p(y) \qquad \left(\text{let } h(y) = \frac{1}{p(y)}\right)$$

$$\frac{dy}{p(y)} = g(x)\, dx$$

$$h(y)\, dy = g(x)\, dx$$

$$\int h(y)\, dy = \int g(x)\, dx$$

$$H(y) = G(x) + C$$

In the last step, we have merged the two constants from indefinite integrals into a single symbol C.

Example Solve $\frac{dy}{dx} = \frac{x}{y}$ with $y(1) = -3$.

Solution:

$$\frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx$$

$$\int y\,dy = \int x\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y^2 = x^2 + 2C$$

$$y = \pm\sqrt{x^2+2C}$$

Plug $x=1$, $y=-3$ (b/c IV is $y(1)=-3$) into the general solution above, we see that the sign must be "$-$" to make $y=-3$.

$$-3 = -\sqrt{1^2+2C} = -\sqrt{1+2C}$$

$$\Leftrightarrow \quad 3 = \sqrt{1+2C}$$

$$9 = 1+2C$$

$$C = 4$$

Notice that we choose the solution with "$-$" sign when we plug in $x=+1$, $y=-3$

Therefore the solution of IVP is $y = -\sqrt{x^2+8}$.

The above example shows sometimes the procedure (for separable ODE) seems to give more than one solution in general for ODE, and when this happens you need to choose one if you need to solve the IVP.

- Constant Solutions

In your algebra days, you may have seen the following equation

$$x(x-2) = 4(x-2) \quad (5)$$

If you just divide both sides by $(x-2)$, you get

$$x = 4.$$

You may think that you have successfully solved equation (5), but eq (5) actually has two solutions: $x=2$ and $x=4$. What's wrong? Because if $x=2$, you are dividing ~~both~~ both sides by $x-2=0$ which is not valid.

Similar things can also happen when solving an ODE.

Example: $y' = e^t\sqrt{1-y^2}$ (Here t is the independent variable)

Solution:

$$\frac{dy}{dt} = e^t\sqrt{1-y^2}$$

$$\frac{dy}{\sqrt{1-y^2}} = e^t\,dt$$

$$\int\frac{dy}{\sqrt{1-y^2}} = \int e^t\,dt$$

$$\sin^{-1}y = e^t + C$$

$$y = \sin(e^t + C)$$

Everything is good provided $\sqrt{1-y^2} \neq 0$ (notice that you divide both sides by $\sqrt{1-y^2}$ to get $\frac{dy}{\sqrt{1-y^2}} = e^t\,dt$)

So what if $\sqrt{1-y^2} = 0$?

This would arise if $y = \pm 1$ and these two are indeed valid solutions to the ODE ($y = 1$ and $y = -1$ are two constant functions of independent variable t)

Therefore ~~we~~ the solutions of the ODE are :

$$y(t) = -1 \ , \quad y(t) = 1 \ , \quad y(t) = \sin(e^t + C) \ .$$

- Overlap :

Just to clear things up, when we classify ODE into different types, one ODE could fall into more than one category :

Example : $\frac{dy}{dx} = xy$ is both separable and first-order linear.

- Justification (Not required to read!)

Suppose $\quad \frac{dy}{dx} = g(x)\, p(y)$

$$h(y(x))\, y'(x) = g(x)$$

Note we assume ($h(y) = \frac{1}{p(y)}$)

$$\int h(y(x))\, y'(x)\, dx = \int g(x)\, dx \qquad (6)$$

On the left hand side, we may use substitution $u = y(x)$ in integration and get

$$\int h(y(x))\, y'(x)\, dx = \int h(u)\, du = H(u) + C_1$$

$u = y(x)$

And thus (6) gives us

$$C_1 + H(y(x)) = G(x) + C_2$$

Of course this implies $H(y(x)) = G(x) + C$ just as our method before.

2.3 Linear Equations

- Linear first-order ODEs can always be written as the form

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = b(x) \qquad (1)$$

Examples :
$$4\frac{dy}{dx} + 5y = 0$$
$$4x\frac{dy}{dx} + e^x y = \sin(x)$$

Divide by $a_1(x)$ in (1) we can get

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}\,y = \frac{b(x)}{a_1(x)} \quad \leftarrow \text{standard form}$$

$$\frac{dy}{dx} + a(x)\,y = b(x) \qquad (2)$$

- Methods of solving Standard Form (2)

Example : $\frac{dy}{dx} + 5y = 2$

Solution :

$$e^{5x}\frac{dy}{dx} + 5y\,e^{5x} = 2e^{5x} \quad \text{Multiply by } e^{5x}$$

$$\frac{d}{dx}\left(e^{5x}y\right) = 2e^{5x}$$

$$\int\left[\frac{d}{dx}\left(e^{5x}y\right)\right]dx = \int 2e^{5x}\,dx$$

$$e^{5x}y = \frac{2}{5}e^{5x} + C$$

$$y = \frac{2}{5} + Ce^{-5x}.$$

The key step above is to multiply the ODE in standard form by e^{5x}, which is $e^{A(x)}$ where $A(x)$ is an antiderivative of $a(x)$.

(any constant C can be chosen for $A(x)$, it doesn't matter)

General Procedure and Formula :

$$\frac{dy}{dx} + a(x)\, y = b(x)$$

$$e^{A(x)} \frac{dy}{dx} + e^{A(x)} a(x)\, y = b(x)\, e^{A(x)}$$

$$\frac{d}{dx}\left(e^{A(x)} y\right) = b(x)\, e^{A(x)}$$

Can be justified if you take derivative of $e^{A(x)} y$ by using product rule and chain rule.

$$e^{A(x)} y = \int b(x)\, e^{A(x)}\, dx$$

$$y = e^{-A(x)} \int b(x)\, e^{A(x)}\, dx$$

So the general solution of standard form (2) is given by

$$y = e^{-A(x)} \underbrace{\int b(x)\, e^{A(x)}\, dx} \qquad (3)$$

Notice the constant C is hiding in $\int b(x)\, e^{A(x)}\, dx$.

Example : $x \frac{dy}{dx} + 2y = x^4$

Solution : First write as standard form (divide by x)

$$\frac{dy}{dx} + \underbrace{\frac{2}{x}}_{a(x)} y = \underbrace{x^3}_{b(x)} .$$

The reason I write quotation marks here is because this indefinite integral, for we do not need constant C.

Use formula (3) we have ($A(x) =$ "$\int \frac{2}{x} dx$" $= 2\ln|x|$)

$$y = e^{-2\ln|x|} \int x^3 e^{2\ln|x|}\, dx .$$

Compute $\int x^3 e^{2\ln|x|}\, dx = \int x^3 |x|^2 dx = \int x^5 dx$

$$= \frac{1}{6} x^6 + C$$

Don't forget the parentheses

$$\Rightarrow y = |x|^{-2} \underbrace{\left(\tfrac{1}{6}x^6 + C\right)} = x^{-2}\left(\tfrac{1}{6}x^6 + C\right) = \frac{1}{6}x^4 + \frac{C}{x^2} .$$

Example: $\begin{cases} y' - 6y = e^t \\ y(0) = 2 \end{cases}$

Solution: $y = e^{-A(t)} \int e^{A(t)} \cdot e^t \, dt$ ← according to (3)

$$A(t) = \int -6 \, dt = -6t$$

So $y = e^{6t} \int e^{-6t} e^t \, dt.$

Since $\int e^{-6t} e^t \, dt = \int e^{-5t} dt = -\frac{1}{5} e^{-5t} + C$

We obtain $y = e^{6t} \left(-\frac{1}{5} e^{-5t} + C \right)$ ← Don't forget the parentheses

$$= -\frac{1}{5} e^t + C e^{6t}$$

Now plugging in the initial value condition $y(0) = 2$:

$$2 = -\frac{1}{5} e^0 + C e^{6 \cdot 0}$$

$$\Rightarrow \quad 2 = -\frac{1}{5} + C \quad \Rightarrow \quad C = \frac{11}{5}$$

The solution of IVP is $y = -\frac{1}{5} e^t + \frac{11}{5} e^{6t}$.

Example: $\frac{1}{x} \frac{dy}{dx} - \frac{2y}{x^2} = x \cos(x)$

Solution: First write as standard form

$$\frac{dy}{dx} + \underbrace{\frac{-2}{x}}_{a(x)} y = \underbrace{\cos(x) x^2}_{b(x)}$$

Compute

$$A(x) = \int \frac{-2}{x} dx = -2 \ln|x| = \ln(|x|^{-2}) = \ln\left(\frac{1}{x^2}\right).$$

$$\int e^{A(x)} b(x) \, dx = \int e^{\ln\left(\frac{1}{x^2}\right)} \cos(x) x^2 \, dx = \int \frac{1}{x^2} \cos(x) x^2 dx = \int \cos(x) dx$$

$$= \sin(x) + C$$

So :
$$y = e^{-A(x)} \int e^{A(x)} b(x)\,dx$$
$$= e^{-\ln\left(\frac{1}{x^2}\right)} \left(\sin(x) + C \right) \leftarrow$$
Don't forget the parentheses !
$$= e^{\ln(x^2)} \left(\sin(x) + C\right)$$
$$= x^2 \left(\sin(x) + C\right)$$

Remark : As we have seen in the examples, calculation of $\int e^{A(x)} b(x) dx$ might be complicated. One needs to first do some simplication before finding the antiderivative, the most common simplification is
$$e^{\ln(f(x))} = f(x)$$
(we use $e^{\ln\left(\frac{1}{x^2}\right)} = \frac{1}{x^2}$ in the example above).

Remark: The constant C in $A(x)$ doesn't affect the final solution.
In the example above, we could choose
$$A(x) = \ln\left(\frac{1}{x^2}\right) + 7,$$
then
$$\int e^{A(x)} b(x)\,dx = \int e^{\ln\left(\frac{1}{x^2}\right) + 7} \cos(x)\, x^2\, dx$$
$$= \int \frac{e^7}{x^2} \cos(x)\, x^2\, dx = \int e^7 \cos(x)\, dx = e^7 \left(\sin(x) + C\right)$$
So
$$y = e^{-\ln\left(\frac{1}{x^2}\right) - 7} \left(e^7 \left(\sin(x) + C\right)\right) = x^2 e^{-7} \cdot e^7 \left(\sin(x) + C\right)$$
$$= x^2 \left(\sin(x) + C\right) \leftarrow$$
the same solution in the end.

2.4 Exact Equations

~~Sometimes~~ the solution of first-order ODE is (can be written) usually in a form

$$F(x, y) = C.$$

Taking derivative with respect to x :

$$\frac{d}{dx} F(x,y) = \boxed{\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0}. \quad (1)$$

Example : $F(x, y) = x^3 y^2 = C$

$$\frac{d}{dx}(x^3 y^2) = 3x^2 y^2 + x^3 \cdot 2y \frac{dy}{dx} = 0.$$

If one can reverse the process before, we can somehow solve ODE (1) and find solution in the form $F(x, y) = C$.

Remark : As in our textbook, (1) may also be written as the form $\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$ ($3x^2y^2 dx + x^3 \cdot 2y\, dy = 0$ in example), this does mean the same thing as (1).

• Detecting Exactness :

Theorem and ~~Def~~ Definition: For an ODE : $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ (2)

or $M(x, y)\,dx + N(x, y)\,dy = 0$ (2'),

the equation is exact if $\exists$ $F(x, y)$ s.t. $M(x,y) = \frac{\partial F}{\partial x}(x, y)$, $N(x, y) = \frac{\partial F}{\partial y}(x, y)$. Definition

This can be detected by checking $\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$ or not. If this holds, then there must exist an F, and the equation is exact.

Way to detect exactness

$$\overset{M(x,y)}{y} + \overset{N(x,y)}{(x+2y)} \frac{dy}{dx} = 0$$

Example :

$$\frac{\partial M}{\partial y} = 1 \ , \quad \frac{\partial N}{\partial x} = 1 \ , \quad \text{so } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \ ,$$

therefore the equation is exact.

• How to find $F(x,y)$?

Still use the example above, $y + (x+2y)\frac{dy}{dx} = 0$.

We want to find $F(x,y)$ satisfying

$$\frac{\partial F}{\partial x} = y \ (3.1), \quad \frac{\partial F}{\partial y} = x + 2y \ (3.2) .$$

From (3.1), view "y" as a coefficient and perform indefinite integral for "x", we get

$$F(x,y) = xy + g(y)$$

Why ? $\int y \, dx = xy + C$?

we have the term $g(y)$ b/c any function only depending on y has zero partial derivative with respect to x.

Now according to (3.2),

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(xy + g(y)\right) = x + g'(y) = x + 2y$$

$$\Rightarrow \quad g'(y) = 2y .$$

So $g(y) = \int 2y \, dy = y^2 + C$

Therefore $F(x,y) = xy + y^2 + C$.

We can choose any constant C, so choose $C=0$ and get

$F(x,y) = xy + y^2$.

Once we have solved $F(x,y)$, we know the general solution of the exact equation is $F(x,y) = C$. ($xy + y^2 = C$ in the example above)

Example: Solve $x+1+\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx} = 0$.

Solution:

$$\underbrace{\left(x+1+\frac{1}{y}\right)}_{M(x,y)} dx + \underbrace{\left(-\frac{x}{y^2}\right)}_{N(x,y)} dy = 0$$

Check whether the equation is exact.

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}, \quad \frac{\partial N}{\partial x} = -\frac{1}{y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, it is exact. Now try to find $F(x,y)$:

$$\frac{\partial F}{\partial x} = x+1+\frac{1}{y} \quad (4.1) \qquad \frac{\partial F}{\partial y} = -\frac{x}{y^2} \quad (4.2)$$

From equation (4.1), by doing indefinite integral with respect to x and view y as a coefficient, we get

$$\underline{F(x,y) = \frac{1}{2}x^2 + x + \frac{x}{y} + g(y)}$$

So $\frac{\partial F}{\partial y} = -\frac{x}{y^2} + g'(y)$ and from (4.2):

$$-\frac{x}{y^2} + g'(y) = -\frac{x}{y^2}$$

$\Rightarrow \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = D$, where D can be any number

Just choose $D=0$, so $g(y)=0$ and

$$F(x,y) = \frac{1}{2}x^2 + x + \frac{x}{y}$$

The solution of the ODE is $\frac{1}{2}x^2 + x + \frac{x}{y} = C$.

This step can be written as

$$F(x,y) = \underline{\int \left(x+1+\frac{1}{y}\right) dx} + g(y) = \frac{1}{2}x^2 + x + \frac{x}{y} + g(y)$$

We didn't put "+C" for this integral b/c "+C" can be included in $g(y)$.

Example : Solve $\dfrac{dy}{dx} = -\dfrac{2xy^2+1}{2x^2y}$

Solution :

$$dy = -\frac{2xy^2+1}{2x^2y}\,dx$$

$$\frac{2xy^2+1}{2x^2y}\,dx + 1\cdot dy = 0 \qquad (5)$$

$$(2xy^2+1)\,dx + 2x^2y\,dy = 0 \qquad (6)$$

In this example, both (5) and (6) are in the form we want:

$$M(x,y)\,dx + N(x,y)\,dy = 0 .$$

However, if we try to use (5), then

$$\frac{\partial}{\partial y}\left(\frac{2xy^2+1}{2x^2y}\right) - \frac{\partial}{\partial x}(1) \neq 0 ,$$

so we find out it is not an exact equation. We don't know how to solve.

If we try to solve from (6),

$$\frac{\partial}{\partial y}(2xy^2+1) - \frac{\partial}{\partial x}(2x^2y) = 4xy - 4xy = 0 .$$

It is exact. Find $F(x,y)$ s.t.

$$\frac{\partial F}{\partial x} = 2xy^2+1 \ (7.2) \qquad \frac{\partial F}{\partial y} = 2x^2y \ (7.3)$$

From (7.2), $\quad F(x,y) = \int (2xy^2+1)\,dx + g(y)$

$$= x^2y^2 + x + g(y)$$

Now using (7.3), $\quad \dfrac{\partial F}{\partial y} = \dfrac{\partial}{\partial y}\left(x^2y^2 + x + g(y)\right)$

$$= 2y\,x^2 + g'(y) = 2x^2y$$

$\Rightarrow g'(y) = 0 \implies$ We can choose $g(y) = 0$

Therefore $F(x, y) = x^2y^2 + x$,

and solution of the ODE is $x^2y^2 + x = C$.

Not required, totally irrelevant to the exam!

Remark: The textbook contains a proof of why we only need to check $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in order to know there exists a function $F(x, y)$ s.t.

$$(*)\begin{cases} \frac{\partial F}{\partial x}(x, y) = M(x, y), \\ \frac{\partial F}{\partial y}(x, y) = N(x, y) \end{cases}$$

The proof is on page 60 "Proof of Theorem 2", it is exactly writing the procedure of computing $F(x, y)$ in an abstract way.

The intuition is that if $F(x, y)$ is smooth, and $(*)$ holds, then from calculus III you know $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$, and thus we must have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. This argument shows that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is always a necessary condition for the existence of $F(x, y)$ satisfying $(*)$.

For the sufficiency, it actually fails when the domain of functions ~~are~~ is not so good. Consider for example,

$$\underbrace{\frac{-y}{x^2+y^2}}_{M(x, y)} dx + \underbrace{\frac{x}{x^2+y^2}}_{N(x, y)} dy = 0$$

Although $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, there is not such a $F(x, y)$ satisfying $(*)$ in the domain $\{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$.

You may think $F(x, y) = \arctan(\frac{y}{x})$ is a choice, but it cannot even be defined continuously in the domain.

$$\frac{2xy^2+1}{2x^2y}dx + 1\,dy = 0$$

2.5 Special Integrating Factors

In the last example ~~of~~ of section 2.4 "$(2xy^2+1)dx + 2x^2y\,dy = 0$"

we see that sometimes even if the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad (1)$$

is not exact, we can make it to be exact by multiplying $\mu(x,y)$ s.t.

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0 \quad (2)$$

is exact. Then this $\mu(x,y)$ is called an integrating factor of the equation (1).

Def of integrating factor: (i) Equation (1) is not exact
(ii) Equation (2) is exact.

<u>Example</u>: Show that $\mu(x,y) = xy^2$ is an integrating factor for $(2y-6x)\,dx + (3x-4x^2y^{-1})\,dy = 0$ (3)

and use this integrating factor to solve the equation.

Solution: First check (3) is not exact.

$$\frac{\partial}{\partial y}(2y-6x) - \frac{\partial}{\partial x}(3x-4x^2y^{-1})$$

$$= 2 - (3 - 8xy^{-1}) \neq 0 \Rightarrow (3) \text{ is not exact.}$$

Now multiply (3) by $\mu(x,y) = xy^2$ to get

$$\underbrace{(2xy^3 - 6x^2y^2)}_{\tilde{M}(x,y)}dx + \underbrace{(3x^2y^2 - 4x^3y)}_{\tilde{N}(x,y)}dy = 0 \quad (4)$$

$$\frac{\partial \tilde{M}}{\partial y} = 2x \cdot 3y^2 - 6x^2 \cdot 2y = 6xy^2 - 12x^2 y$$

$$\frac{\partial \tilde{N}}{\partial x} = 3y^2 \cdot 2x - 4 \cdot 3x^2 y = 6xy^2 - 12x^2 y$$

$\frac{\partial \tilde{M}}{\partial y} = \frac{\partial \tilde{N}}{\partial x} \Rightarrow$ (4) is exact.

Therefore $\mu(x, y)$ is an integrating factor.

Now solve the ODE. Use the exact equation to solve!

Want to find $F(x, y)$ s.t.

$$\frac{\partial F}{\partial x} = 2xy^3 - 6x^2y^2, \quad \frac{\partial F}{\partial y} = 3x^2y^2 - 4x^3y.$$

$$F(x, y) = \int (2xy^3 - 6x^2y^2)\, dx + g(y)$$

$$= x^2y^3 - 2x^3y^2 + g(y)$$

Now we want

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(x^2y^3 - 2x^3y^2 + g(y)\right)$$

$$= x^2 \cdot 3y^2 - 2x^3 \cdot 2y + g'(y)$$

$$= 3x^2y^2 - 4x^3y + g'(y) = 3x^2y^2 - 4x^3y$$

$\Rightarrow g'(y) = 0$. So we can choose $g(y) = 0$

therefore $F(x, y) = x^2y^3 - 2x^3y^2$ and

the solution of ODE is given by

$$x^2y^3 - 2x^3y^2 = C.$$

In general, integrating factors are hard to find. But the following theorem gives us the integrating factor under certain situations:

No need to memorize, will be given if needed in the exam.

__Theorem__: If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N$ only depends on x, then $\mu(x) = \exp\left[\int \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right) dx\right]$ is an integrating factor for equation (1).

If $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)/M$ only depends on y, then

$$\mu(y) = \exp\left[\int \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\right) dy\right]$$

is an integrating factor for equation (1).

__Example__: Solve $\underbrace{(2x^2+y)}_{M(x,y)}\,dx + \underbrace{(x^2y - x)}_{N(x,y)}\,dy = 0$ (5)

Solution:

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 - (2xy-1) = \overset{2-2xy}{\cancel{1-(2xy-1)}} \neq 0$$

So (5) is not exact.

$$\underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{= -\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}/M = \frac{2xy-2}{2x^2+y} \quad \text{not only depends on } y$$

But $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N = \frac{2-2xy}{x^2y - x} = \frac{2(1-xy)}{x(xy-1)}$

$$= \frac{-2}{x} \quad \text{only depends on } x.$$

So consider integrating factor

$$\mu(x) = \exp\left[\int \frac{-2}{x}\,dx\right]$$

$$= \exp\left[-2\ln|x| + C\right]$$

We can choose $C = 0$. It won't affect anything in the end.

$$= \exp\left(\ln\left(|x|^{-2}\right)\right)$$

$$= |x|^{-2} = \frac{1}{x^2}$$

Multiply (5) by $\frac{1}{x^2}$ we get

$$\left(2 + \frac{y}{x^2}\right)dx + \left(y - \frac{1}{x}\right)dy = 0$$

this is exact by the theorem (one can also check).

Try to find $F(x,y)$ s.t. $\frac{\partial F}{\partial x} = 2 + \frac{y}{x^2}$, $\frac{\partial F}{\partial y} = y - \frac{1}{x}$.

$$F(x,y) = \int \left(2 + \frac{y}{x^2}\right)dx + g(y)$$

$$= 2x - \frac{y}{x} + g(y)$$

Now we want

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(2x - \frac{y}{x} + g(y)\right) = -\frac{1}{x} + g'(y) = y - \frac{1}{x}$$

$$\Rightarrow \quad g'(y) = y \quad \Rightarrow \quad g(y) = \int y\,dy = \frac{1}{2}y^2 + C$$

choose to be 0

So we can choose $g(y) = \frac{1}{2}y^2$,

and thus $F(x,y) = 2x - \frac{y}{x} + \frac{y^2}{2}$.

Therefore the solution of the ODE is given by

$$2x - \frac{y}{x} + \frac{y^2}{2} = C.$$

Not required

Remark: Since we are dividing by x^2, we may lose constant solution $x = 0$ (if $x^2=0$ then $x=0$).
If we say y is dependent variable and x is independent variable, then $x=0$ is only true at one point so it doesn't affect anything.
But it's hard to tell from (5) which variable is independent. If we write

$$(2x^2+y) + (x^2y-x)\frac{dy}{dx} = 0,$$

then $x=0$ is not a solution; but if we write

$$(2x^2+y)\frac{dx}{dy} + (x^2y-x) = 0,$$

then $x=0$ is a constant solution.

Example: Find an integrating factor for the equation

$$2y + x\frac{dy}{dx} = 0.$$

2.6 Substitutions and Transformations

Like you have learned in calculus class, substitutions can be used to transform the integral into a simpler form, they can also be used in solving ODEs.

We will learn two substitutions here, each is used to deal with one certain type of ODE. There are other substitutions in the textbook, which are not required in this class, but you can learn them by yourself for interests.

1. Homogeneous Equations

this word comes from the fact that the right hand side of (1) is a function only depending on $\frac{y}{x}$

$$(1) \quad \frac{dy}{dx} = G\left(\frac{y}{x}\right)$$

Examples: Write ODE into form (1).

(i) $\frac{dy}{dx} = \frac{y(\ln y - \ln x)}{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \ln\left(\frac{y}{x}\right) = G\left(\frac{y}{x}\right) \text{ where } G(v) = v \ln v.$$

(ii) $(x-y)dx + x\,dy = 0$

$$\Rightarrow (x-y) + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y-x}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x} - \frac{x}{x}}{\frac{x}{x}} \Rightarrow \frac{dy}{dx} = \frac{y}{x} - 1.$$

key step: divided both numerator and denominator by x^k. Here we choose $k=1$.

(iii) $(xy + y^2 + x^2)dx + (y^2 - x^2)\,dy = 0$

$$\Rightarrow \quad xy + y^2 + x^2 + (y^2 - x^2)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2 - y^2}$$

Divide both numerator and denominator by x^2. We get:

$$\frac{dy}{dx} = \frac{v + v^2 + 1}{1 - v^2} \quad \text{where } v = \frac{y}{x}.$$

<u>How to solve by using substitution $v = \frac{y}{x}$:</u>

If $v = \frac{y}{x}$, then $y = vx$, so

$$\frac{dy}{dx} = \frac{d}{dx}(vx) = \frac{dv}{dx}x + v\frac{d}{dx}(x) = x\frac{dv}{dx} + v.$$

<u>Example</u>: Solve $(x - y)\,dx + x\,dy = 0$ explicitly.

This means the final answer should be expressed as $y = \varphi(x)$. Something like $x^2 + y^2 = C$ is not allowed.

We have shown that this can be written as

$$\frac{dy}{dx} = \frac{y}{x} - 1$$

Use substitution $v = \frac{y}{x}$, then it becomes

$$x\frac{dv}{dx} + v = v - 1$$

$$x\frac{dv}{dx} = -1$$

$$dv = -\frac{1}{x}dx$$

$$v = -\ln|x| + C$$

Plug in $v = \frac{y}{x}$ back: $\frac{y}{x} = -\ln|x| + C$

$$\Rightarrow \quad y = -x\ln|x| + Cx$$

General Procedure for solving $\dfrac{dy}{dx} = G\left(\dfrac{y}{x}\right)$

After substitution $v = \dfrac{y}{x}$, this becomes

$$x\frac{dv}{dx} + v = G(v)$$

$$x\frac{dv}{dx} = G(v) - v \quad \leftarrow \text{Separable ODE}$$

$$\frac{dv}{G(v) - v} = \frac{1}{x}\,dx \qquad (2)$$

Solve the separable ODE for $G(v)$ given in the problem.

Steps: (i) Write ODE into the form (1)

(ii) Use substitution to transform (1) into (2) — A separable ODE

$\hookrightarrow v = \dfrac{y}{x}$

(iii) Solve (2) (But after this, the solution is in terms of x, v)

(iv) Express the answer in terms of original variables x and y.

Example: Solve
$$\begin{cases} (xy + y^2 + x^2)\,dx - x^2\,dy = 0 \\ y(1) = 0 \end{cases}$$

Solution: Write the ODE into the form in (1)

$$\frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1$$

Substitution $v = \dfrac{y}{x}$:

$$x\frac{dv}{dx} + v = v + v^2 + 1$$

$$x\frac{dv}{dx} = v^2+1$$

$$\frac{dv}{v^2+1} = \frac{1}{x}\,dx$$

$$\int \frac{dv}{v^2+1} = \int \frac{1}{x}\,dx$$

$$\arctan(v) = \ln|x| + C$$

So

$$v = \tan(\ln|x| + C)$$

Plug in $v = \frac{y}{x}$ back to the equation above

$$\frac{y}{x} = \tan(\ln|x| + C)$$

$$y = x\tan(\ln|x| + C)$$

To solve the IVP, plug in $x=1$, $y=0$:

$$0 = 1\cdot\tan(\ln|1| + C)$$

$$0 = \tan(C)$$

$\Rightarrow$ $C = k\pi$, k is an integer

(We'll see immediately we can choose $k=0$ here)

So $y = x\underbrace{\tan(\ln|x| + k\pi)}_{=\tan(\ln|x|)}$

$\Rightarrow$ $y = x\tan(\ln|x|)$ (3)

For our IV $y(1)=0$, we are interested in the solution containing $x=1$. Notice that $\ln|x|$ is ~~[illegible]~~ not

Not required in this class

well-defined at $x=0$. It is defined continuously on $(-\infty, 0)$ or $(0, +\infty)$. In a more rigorous way, we should choose the interval containing $x=1$, i.e $(0, +\infty)$, and write the solution as

$$y = x\tan(\ln(x)), \quad x>0. \quad (4)$$

2. ~~Bern~~ Bernoulli Equations

$$(5) \quad \frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n\neq 0, n\neq 1.$$

If $n=0$ or 1, equation (5) is a linear ODE, so we have already learned how to solve it.

Writing ODE into the form in (5) is usually easy!

example: $\dfrac{dx}{dt} + tx^3 + \dfrac{x}{t} = 0$

$$\Rightarrow \quad \frac{dx}{dt} + \frac{1}{t}x = -tx^3, \text{ so } n=3.$$

(Notice here x is the dependent variable and t is the independent variable).

How to solve ODE by using substitution $v = y^{1-n}$

Example: Solve $\dfrac{dy}{dx} - 5y = -\dfrac{5}{2}xy^3$

Divide ~~Multiply~~ by y^n (here $n=3$) to get

$$y^{-3}\frac{dy}{dx} - 5y^{-2} = -\frac{5}{2}x \quad (6)$$

Use substitution $v = y^{-2}$, then

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$$

So (6) becomes

$$\frac{1}{-2}\frac{dv}{dx} - 5v = -\frac{5}{2}x \quad \Leftarrow \text{linear ODE}$$

Now we get a linear ODE and we can solve it:

$$\frac{dv}{dx} + 10v = 5x$$

$$e^{10x}\frac{dv}{dx} + 10e^{10x}v = 5xe^{10x}$$

$$\frac{d}{dx}(ve^{10x}) = 5xe^{10x}$$

$$ve^{10x} = \int 5xe^{10x}$$

$$ve^{10x} = \frac{1}{2}xe^{10x} - \frac{1}{20}e^{10x} + C$$

$$v = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}.$$

Substitute back $v = y^{-2}$:

$$y^{-2} = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}.$$

Notice we divide by y^3 at the beginning which is only correct when $y^3 \neq 0$. If $y^3 = 0$, i.e. $y = 0$ one can check this is also a solution. So the solutions of the ODE are:

$$y = 0, \quad y^{-2} = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}.$$

General Procedure for solving $\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$

dividing by y^n: $\quad y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x) \quad (7)$

Use substitution $v = y^{1-n}$,

$$\frac{dv}{dx} = (1-n)\,y^{-n}\,\frac{dy}{dx}.$$

So (7) becomes

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)\,v = Q(x) \quad \leftarrow \text{linear ODE}$$

Solve it and express the final answer in terms of x and y.

Example: $\quad \frac{dy}{dx} - y = e^{2x}y^3$

Solution: Bernoulli equation, $n = 3$.

Dividing by y^3: $\quad y^{-3}\frac{dy}{dx} + (-y^{-2}) = e^{2x} \quad (8)$

Substitution $v = y^{1-3} = y^{-2}$.

$$\frac{dv}{dx} = -2\,y^{-3}\,\frac{dy}{dx}$$

So (8) becomes

$$-\frac{1}{2}\frac{dv}{dx} - v = e^{2x}$$

$$\frac{dv}{dx} + 2v = -2e^{2x}$$

$$e^{2x}\frac{dv}{dx} + 2e^{2x}v = -2e^{2x+2x}$$

$$\frac{d}{dx}\left(e^{2x}v\right) = -2e^{4x}$$

$$e^{2x} v = -\frac{1}{2} e^{4x} + C$$

$$v = -\frac{1}{2} e^{2x} + Ce^{-2x} \Rightarrow y^{-2} = -\frac{1}{2} e^{2x} + Ce^{-2x}.$$

Since we are dividing by y^3, we also need to check $y^3 = 0$ i.e. $y = 0$.

It can be verified that $y=0$ is a solution of the ODE.

So in conclusion, we have solutions

$$y = 0, \quad y^{-2} = -\frac{1}{2} e^{2x} + Ce^{-2x}.$$

Do not forget constant solution $y=0$ for Bernoulli equation.