MATH 231. EXAM 3

Name:____

Instructions: This is a closed-book exam, and calculators can only be used to do basic arithmetic operations (not allowed for differentiation and integration). The last page contains a table for integrals and some results from textbook, which might be helpful. Read each problem carefully. You must show your work to receive credit. Partial credit will be given for any work relevant to the problem.

Problem	**Grade**
1	
2	
3	
4	
5	
Total	

There is a total of 100 points.

Problem 1 (15 points): Determine whether the critical point $(0,0)$ in the following system is stable or not. (**Hint**: pay attention to the order of x and y on the right hand sides.)

$$\begin{cases} \dfrac{dx}{dt} = -2x + 2y \\ \dfrac{dy}{dt} = y + 2x. \end{cases}$$

Problem 2 (20 points): (a) (10 points) Compute $\mathscr{L}[f(t)]$ where

$$f(t) = \begin{cases} t, & 0 \le t < 5, \\ 0, & t \ge 5. \end{cases}$$

(b) (10 points) Compute $\mathscr{L}^{-1}[F(s)]$ where $F(s) = \dfrac{s}{s^2 - 4s + 5}$

Problem 3 (20 points): Use the **Laplace transform** to solve the IVP:

$$y'' - 3y' + 2y = 2e^{2t} \quad \text{with} \quad y(0) = 0, y'(0) = 1.$$

Problem 4 (20 points): Solve the IVP:

$$y' + y = f(t) \quad \text{with} \quad y(0) = -1.$$

where

$$f(t) = \begin{cases} -1, & 0 \leq t < 1, \\ 1, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$$

Problem 5 (25 points): Find the recurrence relation and the first four nonzero terms in a power series expansion about $x = 0$ for the solution to the following IVP:

$$y'' + (x+1)y' - y = 0 \quad \text{with} \quad y(0) = 4,\ y'(0) = -2.$$

Brief Table for Integrals:

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x + \sqrt{x^2+a^2}|.$ $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x + \sqrt{x^2-a^2}|, \quad x^2 \geq a^2.$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin(\frac{x}{a}), \quad a^2 \geq x^2.$ $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan(\frac{x}{a}).$

$\int \tan(x)dx = -\ln|\cos x|.$ $\int \cot(x)dx = \ln|\sin x|.$

$\int \sec(x)dx = \ln|\sec x + \tan x|.$ $\int \csc(x)dx = -\ln|\csc x + \cot x|.$

Common trigonometric substitutions:

(1) For integrand involving $\sqrt{a^2-x^2}$, set $x = a\sin(\theta)$,
(2) For integrand involving $\sqrt{a^2+x^2}$, set $x = a\tan(\theta)$,
(3) For integrand involving $\sqrt{x^2-a^2}$, set $x = a\sec(\theta)$,
(4) For $\int \tan^n(x)\sec^{2m}(x)dx$, set $u = \tan(x)$,
(5) For $\int \cot^n(x)\csc^{2m}(x)dx$, set $u = \cot(x)$.

Brief Table of Laplace Transforms

$f(t)$	$\mathscr{L}[f(t)] = F(s)$	$f(t)$	$\mathscr{L}[f(t)] = F(s)$
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
$t^n,\ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$	$e^{at}t^n,\ n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$
$\sin bt$	$\frac{b}{s^2+b^2}$	$\cos bt$	$\frac{s}{s^2+b^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$

Properties of Laplace Transforms

$$\mathscr{L}[f+g] = \mathscr{L}[f] + \mathscr{L}[g]$$

$$\mathscr{L}[cf] = c\mathscr{L}[f] \qquad \text{for any constant } c$$

$$\mathscr{L}\left[e^{at}f(t)\right](s) = \mathscr{L}[f](s-a)$$

$$\mathscr{L}[f'](s) = s\mathscr{L}[f](s) - f(0)$$

$$\mathscr{L}[f''](s) = s^2\mathscr{L}[f](s) - sf(0) - f'(0)$$

$$\mathscr{L}[t^n f(t)](s) = (-1)^n \frac{d^n}{ds^n}\left(\mathscr{L}[f](s)\right)$$

$$\mathscr{L}[f(t-a)u(t-a)](s) = e^{-as}F(s), \text{ where } F(s) = \mathscr{L}[f(t)]$$

$$\mathscr{L}^{-1}\left[e^{-as}F(s)\right] = f(t-a)u(t-a), \text{ where } f(t) = \mathscr{L}^{-1}[F(s)]$$

$$\mathscr{L}[g(t)u(t-a)](s) = e^{-as}\mathscr{L}[g(t+a)](s)$$