中国科学院大学 2015 秋季学期微积分 III-A01 习题 8

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作业 1. 欧拉公式

$$E := \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}.$$

a) 证明:

$$E^2 = \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) \Gamma\left(\frac{n-k}{n}\right).$$

(b) 验证:

$$E^2 = \frac{\pi^{n-1}}{\sin\frac{\pi}{n} \cdot \cdots \cdot \sin\frac{(n-1)\pi}{n}}$$

(c) 根据恒等式

$$\frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1} (2 - e^{i\frac{2k\pi}{n}}),$$

当 $z \to 1$ 时, 得到下面关系式

$$n = \prod_{k=1}^{n-1} (1 - e^{i\frac{2k\pi}{n}}),$$

从它又得到关系式

$$n = 2^{n-1} \prod_{k=1}^{n-1} \sin\frac{k\pi}{n}.$$

(d) 试利用最后的等式推出欧拉公式.

作业 2. 勒让德公式

$$\Gamma(\alpha)\Gamma\left(\alpha + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2\alpha-1}} \Gamma(2\alpha).$$

(a) 证明

$$B(\alpha, \alpha) = 2\int_0^{\frac{1}{2}} \left(\frac{1}{4} - \left(\frac{1}{2} - x\right)^2\right)^{\alpha-1} dx.$$

(b) 在上述积分中作变量替换, 证明:

$$B(\alpha, \alpha) = \frac{1}{2^{2\alpha-1}} B\left(\frac{1}{2}, \alpha\right).$$

(c) 推出勒让德公式.

作业 3. 拉比积分

$$\int_0^1 \ln\Gamma(x)dx.$$

证明:

a) $\int_0^1 \ln\Gamma(x)dx = \int_0^1 \ln\Gamma(1-x)dx$.

b) $\int_0^1 \ln\Gamma(x)dx = \frac{1}{2}\ln\pi - \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \ln\sin x dx$.

c) $\int_0^{\frac{\pi}{2}} \ln\sin x dx = \int_0^{\frac{\pi}{2}} \ln\sin 2x - \frac{\pi}{2}\ln 2$.

d) $\int_0^{\frac{\pi}{2}} \ln\sin x dx = -\frac{\pi}{2}\ln 2$.

e) $\int_0^1 \ln\Gamma(x)dx = \ln\sqrt{2\pi}$.

作业 4. a) 验证卷积德结合律: $u*(v*w) = (u*v)*w$.

b) 照例设 $\Gamma(\alpha)$ 是欧拉 Γ 函数, $H(x)$ 是赫维赛德函数, 令

$$H_\lambda^\alpha(x) := H(x)\frac{x^{\alpha-1}}{\Gamma(\alpha)}e^{\lambda x}, \qquad \alpha > 0,\ \lambda \in \mathbb{C}.$$

证明: $H_\lambda^\alpha(x) * H_\lambda^\beta(x) = H_\lambda^{\alpha+\beta}(x)$.

c) 验证: 函数 $F = H(x)\frac{x^{n-1}}{(n-1)!}e^{\lambda x}$ 是函数 $f = H(x)e^{\lambda x}$ 的 n 次幂卷积.

作业 5. 称函数 $A(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ 为数列 $a_0, a_1, \cdots$ 的生成函数. 给定两个数列 $\{a_k\}, \{b_k\}$. 如果认为当 $k < 0$ 时, $a_k = b_k = 0$, 那么, 自然地把 $\{a_k\}$ 与 $\{b_k\}$ 的卷积定义作 $\left\{c_k = \sum\limits_m a_m b_{k-m}\right\}$. 试证, 两个数列卷积的生成函数等于它们的生成函数的乘积.

作业 6. 设 $\{\Delta_\alpha, \alpha > 0\}$ 是关于 $\alpha \to 0$ 的 δ-型的函数族. 对 $\alpha > 0$, 定义: $\bar{\Delta}_\alpha = \Delta_\alpha * \Delta_\alpha$. 请证明: 函数族 $\{\bar{\Delta}_\alpha, \alpha > 0\}$ 是关于 $a \to 0$ 的 δ-型函数族.

解答作业 1. a)

$$E^2 = \left(\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right)\right)^2 = \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right)\Gamma\left(\frac{n-k}{n}\right).$$

b) 根据余元公式

$$E^2 = \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right)\Gamma\left(\frac{n-k}{n}\right) = \prod_{k=1}^{n-1} \frac{\pi}{\sin\frac{n\pi}{n}} = \frac{\pi^{n-1}}{\sin\frac{\pi}{n}\cdot\cdots\cdot\sin\frac{(n-1)\pi}{n}}$$

c)

$$n = \lim_{z\to 1}\frac{nz^{n-1}}{1} = \lim_{z\to 1}\frac{z^n-1}{z-1} = \lim_{z\to 1}\prod_{k=1}^{n-1}(2-e^{i\frac{2k\pi}{n}}) = \prod_{k=1}^{n-1}(1-e^{i\frac{2k\pi}{n}}).$$

又有

$$1-e^{i\frac{2k\pi}{n}} = 1-\cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n} = 2\sin^2\frac{k\pi}{n} - 2i\sin\frac{k\pi}{n}\cos\frac{k\pi}{n}.$$

因此,

$$\begin{aligned}
&\prod_{k=1}^{n-1}(1-e^{i\frac{2k\pi}{n}}) = 2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}\prod_{k=1}^{n-1}\left(\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n}\right)\\
= \quad & 2^{n-1}\prod_{k=1}^{n-1}\left(\sin\frac{k\pi}{n}\cdot e^{-i\frac{k\pi}{n}}\cos\frac{k\pi}{n}\right) = 2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}
\end{aligned}$$

d) 由 b) 和 c) 立得.

解答作业 2. a)

$$B(\alpha,\alpha) = \int_0^1 x^{\alpha-1}(1-x)^{\alpha-1}dx = 2\int_0^{\frac{1}{2}}(x-x^2)^{\alpha-1}dx = 2\int_0^{\frac{1}{2}}\left(\frac{1}{4}-\left(\frac{1}{2}-x\right)^2\right)^{\alpha-1}dx.$$

b)

$$\begin{aligned}
B(\alpha,\alpha) &= \frac{2}{4^{\alpha-1}}\int_0^{\frac{1}{2}}(1-(1-2x)^2)^{\alpha-1}dx = \frac{1}{2^{2\alpha-3}}\int_0^1(1-t)^{\alpha-1}\cdot\left(-\frac{1}{4}t^{-\frac{1}{2}}\right)dt\\
&= \frac{1}{2^{2\alpha-1}}\int_0^1 t^{-\frac{1}{2}}(1-t)^{\alpha-1}dt = \frac{1}{2^{2\alpha-1}}B\left(\frac{1}{2},\alpha\right).
\end{aligned}$$

c)

$$\begin{aligned}
&\frac{\Gamma(\alpha)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)} = B\left(\frac{1}{2},\alpha\right), \qquad \frac{\Gamma(\alpha)\Gamma(\alpha)}{\Gamma(2\alpha)} = B(\alpha,\alpha),\\
&\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\cdot\Gamma(2\alpha)\cdot B(\alpha,\alpha)\cdot\frac{1}{B\left(\frac{1}{2},\alpha\right)} = \frac{\sqrt{\pi}}{2^{2\alpha-1}}\Gamma(2\alpha).
\end{aligned}$$

解答作业 3. a) $\int_0^1 \ln\Gamma(x)dx = \int_0^1 \ln\Gamma(1-x)d(1-x) = \int_0^1 \ln\Gamma(1-x)dx$.

b) $\int_0^1 \ln\Gamma(x)dx = \frac{1}{2}\left(\ln\Gamma(x)+\ln\Gamma(1-x)\right)dx = \frac{1}{2}\int_0^1 \ln\left(\frac{\pi}{\sin\pi x}\right)dx = \frac{1}{2}\int_0^1 \ln\pi - \ln\sin\pi dx = \frac{1}{2}\ln\pi - \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\ln\sin x dx$.

c) $\int_0^{\frac{\pi}{2}} \ln\sin 2x - \int_0^{\frac{\pi}{2}} \ln\sin x dx = \int_0^{\frac{\pi}{2}} (\ln 2 + \ln\cos x)\, dx = \frac{\pi}{2}\ln 2 + \int_0^{\frac{\pi}{2}} \ln\sin x dx$.
d) $\int_0^{\frac{\pi}{2}} \ln\sin 2x - \int_0^{\frac{\pi}{2}} \ln\sin x dx = \frac{1}{2}\int_0^{\pi} \ln\sin t dt - \int_0^{\frac{\pi}{2}} \ln\sin x dx = 0$.
e) 由 b) 和 d) 直接可得.
注: $x=0$ 和 $x=1$ 是瑕点, 应该仔细讨论瑕点处的情况, 当然并不影响结果.

解答作业 4. a)

$$\begin{aligned} u*(v*w)(x) &= \int_R u(y)(v*w)(x-y)dy = \int_R u(y)\int_R v(z)w(x-y-z)dzdy \\ &= \int_R u(y)dy\int_R v(z-y)w(x-z)dz; \\ (u*v)*w(x) &= \int_R (u*v)(y)w(x-y)dy = \int_R w(x-y)dy\int_R u(z)v(y-z)dz \\ &= \int_R u(z)dz\int_R v(y-z)w(x-y)dy. \end{aligned}$$

结合上面两式, 证毕.

b)

$$\begin{aligned} H_\lambda^\alpha(x)*H_\lambda^\beta(x) &= \int_R H(y)\frac{y^{\alpha-1}}{\Gamma(\alpha)}e^{\lambda y}H(x-y)\frac{(x-y)^{\beta-1}}{\Gamma(\beta)}e^{\lambda(x-y)}dy \\ &= \int_R H(y)H(x-y)\cdot\frac{y^{\alpha-1}(x-y)^{\beta-1}}{B(\alpha,\beta)\Gamma(\alpha+\beta)}e^{\lambda}xdy \end{aligned}$$

当 $x<0$ 时, 若 $y<0$, 则 $H(y)=0$, 若 $y\geq 0$, 则 $H(x-y)=0$, 因此 $H_\lambda^\alpha(x)*H_\lambda^\beta(x)=H_\lambda^{\alpha+\beta}(x)=0$.
当 $x\geq 0$ 时,

$$\begin{aligned} H_\lambda^\alpha(x)*H_\lambda^\beta(x) &= \int_0^x \frac{y^{\alpha-1}(x-y)^{\beta-1}}{B(\alpha,\beta)\Gamma(\alpha+\beta)}e^{\lambda}xdy = \int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}x^{\alpha+\beta-1}}{B(\alpha,\beta)\Gamma(\alpha+\beta)}e^{\lambda}xdt \\ &= \frac{B(\alpha,\beta)e^{\lambda x}x^{\alpha+\beta-1}}{B(\alpha,\beta)\Gamma(\alpha+\beta)} = H_\lambda^{\alpha+\beta}(x). \end{aligned}$$

综上两种情况, 证毕.

(c) 利用数学归纳法及 (b) 易证.

解答作业 5.

$$\begin{aligned} A(x)B(x) &= \left(\sum_{n=0}^{\infty} a_n x^n\right)\cdot\left(\sum_{n=0}^{\infty} b_n x^n\right); \\ C(x) &= \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n. \end{aligned}$$

$A(x)B(x)$, $C(x)$ 均为多项式函数, 考察每项中 x^n 的系数均为 $\sum_{k=0}^{n} a_k b_{n-k}$. 证毕.

如果要证卷积: 亦可考虑生成函数的卷积等于卷积的生成函数.

解答作业 6. (1) $\bar{\Delta}_\alpha(x)=\int_R \Delta_\alpha(y)\Delta_{\alpha+\beta}(x-y)dy$, 由于 $\Delta_\alpha(y)\geq 0$, $\Delta_\alpha(x-y)\geq 0$, 因此 $\bar{\Delta}_\alpha(x)\geq 0$.

(2)

$$\begin{aligned}\int_R \bar{\Delta}_\alpha(x) &= \int_R\int_R \Delta_\alpha(y)\Delta_\alpha(x-y)dydx = \int_R \Delta_\alpha(y)dy\int_R \Delta_\alpha(x-y)d(x-y)\\ &= \int_R \Delta_\alpha(y)dy = 1.\end{aligned}$$

(3) 对任何一个不包含 0 的邻域 U', 存在包含 0 的邻域 U'', 记 $U_0 = U' + U''$. 进而有

$$\lim_{\alpha\to 0}\int_{U'} \Delta_\alpha(x)dx = \lim_{\alpha\to 0}\int_{U_0} \Delta_\alpha(x)dx - \lim_{\alpha\to 0}\int_{U''} \Delta_\alpha(x)dx = 1-1 = 0.$$

对任意包含 0 的邻域 $[a,b]$, 我们有

$$\begin{aligned}&\lim_{\alpha\to 0}\int_a^b\int_R \Delta_\alpha(y)\Delta_\alpha(x-y)dydx = \lim_{\alpha\to 0}\int_R\int_a^b \Delta_\alpha(y)\Delta_\alpha(x-y)dxdy\\ =\ &\lim_{\alpha\to 0}\int_R \Delta_\alpha(y)dy\int_a^b \Delta_\alpha(x-y)d(x-y)\\ =\ &\lim_{\alpha\to 0}\int_a^b \Delta_\alpha(y)dy\int_a^b \Delta_\alpha(x-y)d(x-y) + \lim_{\alpha\to 0}\int_{R\setminus[a,b]} \Delta_\alpha(y)dy\int_a^b \Delta_\alpha(x-y)d(x-y)\\ =\ &\lim_{\alpha\to 0}\int_a^b \Delta_\alpha(y)dy + 0 = 1.\end{aligned}$$

证毕.