中国科学院大学 2015 春季学期微积分 II-A01 习题 9

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2015 年 5 月 9 日，8:00-9:40

作业 1. (a) 试证: 如果集合 $X \subset \mathbb{R}^n$ 使得 $\mu(X) = 0$, 那么对于这个集的闭包 $\bar{X}$ 也成立等式 $\mu(\bar{X}) = 0$.

(b) 试举出有界的勒贝格零测度集 X, 使 X 的闭包 $\bar{X}$ 不是勒贝格零测度集的例子.

作业 2. 如果有界集 X 不是容许集 (约当可测集), 那么由

$$\int_X f(x)dx = \int_I f\chi_X(x)dx$$

定义积分是否存在?

作业 3. 设 X 是勒贝格零测度集, 而 $f : X \to \mathbb{R}$ 是 X 上的连续有界函数, f 是否总在 X 上可积?

作业 4. 设 X 是具有非零测度的约当可测集, 而 $f : X \to \mathbb{R}$ 是 X 上的连续非负可积函数, $M = \sup\limits_{x \in X} f(x)$. 试证:

$$\lim_{n\to\infty} \left(\int_X f^n(x)dx\right)^{\frac{1}{n}} = M.$$

作业 5. 如果 $f, g \in \mathcal{R}(X)$, 那么成立 Hölder 不等式

$$\left|\int_X (f \cdot g)(x)dx\right| \le \left(\int_X |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_X |g(x)|^q dx\right)^{\frac{1}{q}},$$

其中 $p \ge 1$, $q \ge 1$, 且 $\frac{1}{p} + \frac{1}{q} = 1$.

作业 6. 设 X 是 $\mathbb{R}^n$ 中的约当可测集, 而 X 上的可积函数 $f : E \to \mathbb{R}$ 在它的内点 $a \in X$ 连续, 则

$$\lim_{\delta\to 0^+} \frac{1}{\mu(U_X^\delta(a))} \int_{U_X^\delta(a)} f(x)dx = f(a),$$

其中 $U_X^\delta(a)$ 表示点 a 在集 X 中的 δ-邻域.

祝大家期中考试顺利!

解答作业 1. (a) Firstly, $X = \mathrm{int}(X) \cup \mathrm{bd}(X)$, hence

$$\mu(\bar{X}) = \int_{\bar{X}} 1dx \leq \int_{\mathrm{int}(X)} 1dx + \int_{\mathrm{bd}(X)} 1dx \leq \int_X 1dx + \int_{\mathrm{bd}(X)} 1dx = \mu(X) + \mu(\mathrm{bd}(X)). \tag{1}$$

On the other hand, $\mathrm{bd}(X)$ is compact, and hence it is Jordan measurable and $\mu(\mathrm{bd}(X)) = 0$. Combining with (1), we immediately obtain that $\mu(\bar{X}) = 0$.

(b) Let $X = \{y \mid y \in \mathbb{Q}\} \cap [0,1]$. Then $\bar{X} = [0,1]$.

解答作业 2. Not necessarily exists, for instance, $f(x) = 1$, $X = \{y \mid y \in \mathbb{Q}\} \cap [0,1]$.

解答作业 3. Not necessarily integrable, for instance, $f(x) = 1$, $X = \{y \mid y \in \mathbb{Q}\} \cap [0,1]$.

解答作业 4. For any $n \in \mathbb{N}$, we have $\int_X |f(x)|^n dx \leq \int_X M^n dx = M^n \mu(I)$. Hence, it holds that

$$\left(\int_X |f(x)|^n dx \right)^{\frac{1}{n}} \leq (M^n \mu(I))^{\frac{1}{n}}.$$

Taking limit $n \to \infty$ in both sides of the above inequality, we obtain

$$\limsup_{n\to\infty} \left(\int_X |f(x)|^n dx \right)^{\frac{1}{n}} \leq M. \tag{2}$$

On the other side, denote $y \in X$ satisfying $f(y) = M$. For any $\epsilon > 0$, there exists $\delta > 0$ satisfying

$$|f(x)| \geq M - \epsilon, \qquad \forall\, x \in U(y,\delta) \subset [a,b].$$

Hence

$$\left(\int_X |f(x)|^n dx \right)^{\frac{1}{n}} \geq ((M-\epsilon)^n \mu(U(y,\delta)))^{\frac{1}{n}}.$$

Taking limit $n \to \infty$ in both sides of the above inequality, we obtain

$$\liminf_{n\to\infty} \left(\int_X |f(x)|^n dx \right)^{\frac{1}{n}} \geq \sqrt{M-\epsilon}.$$

Due to the arbitrariness of ϵ, we obtain

$$\liminf_{n\to\infty} \left(\int_X |f(x)|^n dx \right)^{\frac{1}{n}} \geq \sqrt{M}. \tag{3}$$

Combining (2) with (3), and using the Squeeze Theorem, we finish the proof.

解答作业 5. First, we prove the Young's inequality first. For any a, b, p, q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$ab = e^{\log a} e^{\log b} = e^{\frac{1}{p}\log a^p + \frac{1}{q}\log b^q} \leq \frac{1}{p} e^{\log a^p} + \frac{1}{q} e^{\log b^q} = \frac{a^p}{p} + \frac{b^q}{q},$$

where the inequality results from the convexity of e^x.

If $\int_X |f|^p(x)dx = 0$, then f is zero almost everywhere, and the product fg is zero almost everywhere, hence the left-hand side of Hölder's inequality is zero. The same is true if $\int_X |g|^q(x)dx = 0$.

Therefore, we may assume $\int_X |f|^p(x)dx > 0$ and $\int_X |g|^q(x)dx > 0$ in the following.

Dividing f and g by $\left(\int_X |f|^p(x)dx\right)^{\frac{1}{p}}$ and $\left(\int_X |g|^q(x)dx\right)^{\frac{1}{q}}$, respectively, we can assume that $\left(\int_X |f|^p(x)dx\right)^{\frac{1}{p}} = \left(\int_X |g|^q(x)dx\right)^{\frac{1}{q}} = 1$.

We now use Young's inequality,

$$|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}, \qquad \forall\, x \in X.$$

Integrating both sides gives

$$\left|\int_X |f \cdot g|(x)dx\right| \leq \frac{1}{p} + \frac{1}{q} = 1,$$

which implies the claim.

解答作业 6. First, for any $\delta > 0$, $\overline{U_X^\delta(a)}$ is compact, which implies $\overline{U_X^\delta(a)}$ is Jordan measurable, and hence,

$$\begin{aligned} \mu(U_X^\delta(a)) &= \mu(\overline{U_X^\delta(a)}); \\ \int_{U_X^\delta(a)} f(x)dx &= \int_{\overline{U_X^\delta(a)}} f(x)dx. \end{aligned}$$

On the other hand, combining the compactness of $\overline{U_X^\delta(a)}$ with the continuity of f, we know that f is Lipschitz continuous on $\overline{U_X^\delta(a)}$. Let L_a be the Lipschitz constant with respect to a fixed δ'. Namely, it always holds that

$$|f(x) - f(y)| \leq L_{(a,\delta)}|x-y|, \qquad \forall\, x, y \in \overline{U_X^{\delta'}(a)},$$

which implies

$$f(a) - L_{(a,\delta)}\delta \leq f(a) + L_{(a,\delta)}|x-a| \leq f(x) \leq f(a) + L_{(a,\delta)}|x-a| \leq f(a) + L_{(a,\delta)}\delta$$

holds for any $x \in \overline{U_X^\delta(a)}$, where $\delta \in (0, \delta']$. Then, for any $\delta \in (0, \delta']$, we have

$$\frac{1}{U_X^\delta(a)} \int_{U_X^\delta(a)} f(x)dx = \frac{1}{\overline{U_X^\delta(a)}} \int_{\overline{U_X^\delta(a)}} f(x)dx \leq \frac{1}{\overline{U_X^\delta(a)}} \int_{\overline{U_X^\delta(a)}} f(a) + L_{(a,\delta)}\delta dx = f(a) + L_{(a,\delta)}\delta.$$

Taking limit $\delta \to 0^+$ in both sides of the above inequality, we obtain

$$\limsup_{\delta \to 0^+} \frac{1}{\mu(U_X^\delta(a))} \int_{U_X^\delta(a)} f(x)dx \leq f(a).$$

The other direction can be obtained in the same manner. We finish the proof.