中国科学院大学 2015 春季学期微积分 II-A01 习题 8

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作业 1. 设函数 $f(x), x \in \mathbb{R}$ 在区间 $[1,2]$ 上黎曼可积, 函数 $g(y), y \in \mathbb{R}$ 在区间 $[4,5]$ 上黎曼可积. 证明函数 $h(x,y) = f(x)g(y), (x,y) \in \mathbb{R}^2$ 在区间 $[1,2] \times [4,5]$ 上黎曼可积.

作业 2. 证明零测度集合没有内点.

作业 3. a) 在 $\mathbb{R}^n$ 中构造与狄利克雷函数类似的函数, 并证明, 有界函数 $f : I \to \mathbb{R}$ 几乎在区间 I 的所有点等于零, 但这并不意味着 $f \in \mathcal{R}(I)$.

b) 试证, 如果 $f \in \mathcal{R}(I)$ 且几乎在区间 I 的所有点 $f(x) = 0$, 则

$$\int_I f(x)dx = 0.$$

作业 4. 设 $f(x)$ 在 n 维区间 $I = [a,b]$ $(b_i > a_i,\ i = 1, \cdots, n)$ 上黎曼可积, 且 $\int_I f(x)dx > 0$. 证明一定存在一个小的 n 维区间 $\hat{I} = [c,d] \subset I$ 使得 $c_i < d_i$ $(i = 1, \cdots, n)$ 而且对一切 $x \in [c,d]$ 都有 $f(x) > 0$.

作业 5. 设 $a > 0$, $I = [-a,a] \times [-a,a] \subset \mathbb{R}^2$. 证明

$$\int_I \sin(x+y)dxdy = 0.$$

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测验 1. (3 分) 证明: 函数

$$R(x,y) := \begin{cases} \frac{1}{\max\{n_x, n_y\}}, & \text{如果} x \in \mathbb{Q},\ y \in \mathbb{Q}, \\ & \text{并且} n_x = \min\left\{n \mid x = \frac{m}{n},\ m \in \mathbb{Z},\ n \in \mathbb{N}\right\},\ n_y = \min\left\{n \mid y = \frac{m}{n},\ m \in \mathbb{Z},\ n \in \mathbb{N}\right\}; \\ 0, & \text{otherwise}. \end{cases}$$

黎曼可积. 并说明当定义中的 max 取 min 时, 对应的函数不可积.

测验 2. (2 分) 证明: 有界函数 $f \in \mathcal{R}(I)$, 这里 $I \subset \mathbb{R}^n$. 当且仅当对于任意的 $\varepsilon > 0$ 和 $\delta > 0$, 存在区间 I 的一个分划 P, 使函数在其上的振幅大于 ε 的那些区间的体积之和不超过 δ.

解答作业 1. Since $f \in \mathcal{R}[1,2]$ and $g \in \mathcal{R}[4,5]$, they are bounded in the corresponding intervals. Namely, there exist $M_1 > 0$ and $M_2 > 0$ such that $|f(x)| < M_1$ for any $x \in [1,2]$ and $|g(x)| < M_2$ for any $x \in [4,5]$. Moreover, for any $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that for any P_1, a partition of $[1,2]$, and P_2, a partition of $[4,5]$, satisfying $\lambda(P_1) < \delta_1$ and $\lambda(P_2) < \delta_2$, it holds

$$\begin{aligned} \sum_{i=1}^{m} \omega(f; I_i^{P_1})|I_i^{P_1}| &< \frac{\epsilon}{2M_2}, \\ \sum_{i=1}^{m} \omega(f; I_i^{P_2})|I_i^{P_2}| &< \frac{\epsilon}{2M_1}. \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, suppose P is a partition of $[1,2] \times [4,5]$ satisfying $\lambda(P) < \delta$. For convenience, let $I_i^P = I_i^{P_1} \times I_i^{P_2}$ $(i = 1, \cdots, m)$. Clearly, we have $|I_i^{P_1}| < \lambda(P) \le \delta_1$ and $|I_i^{P_2}| < \lambda(P) \le \delta_2$ for any $i = 1, \cdots, m$. Then there holds

$$\begin{aligned} & \sum_{i=1}^{m} \omega(h; I_i^P)|I_i^P| = \sum_{i=1}^{m} \omega(f(x)g(y); I_i^P)|I_i^P| \\ \le & \sum_{i=1}^{m} \max\{\omega(f(x)M_2; I_i^{P_1})|I_i^{P_1}|, \omega(M_1 g(y); I_i^{P_2})|I_i^{P_2}|\} \le \epsilon. \end{aligned}$$

We complete the proof.

解答作业 2. Suppose $x \in \mathbb{R}^n$ is an interior point of a zero measure set Ω. There exists a neighborhood of x, denoted as $U_\delta(x)$ for convenience, satisfying $U_\delta(x) \subset \Omega$. Clearly, the interval $[a,b]$, where $a = \left(x_1 - \frac{\sqrt{n}}{n}, \cdots, x_n - \frac{\sqrt{n}}{n}\right)^\top$, $b = \left(x_1 + \frac{\sqrt{n}}{n}, \cdots, x_n + \frac{\sqrt{n}}{n}\right)^\top$, is a subset of $U_\delta(x)$. Meanwhile, we have $a < b$, which implies that $\Omega \supset [a,b]$ is not a zero measure set.

解答作业 3. a) Let

$$f(x) := \begin{cases} 1, & \text{if}\, x_i \in \mathbb{Q},\ \forall i = 1, \cdots, n; \\ 0, & \text{otherwise.} \end{cases}$$

Without loss of generality, we only consider $I = [0,1]^n$. For any $\delta > 0$, let $p := \max\{2, \left[\frac{1}{\delta}\right]\}$, then it holds $p\delta > 1$. We consider the the following partition

$$I_i := \prod_{j=1}^{n} \left[\frac{i_j - 1}{p-1}, \frac{i_j}{p-1}\right],$$

where

$$i \in \Omega := \{(i_1, \cdots, i_n) \mid i_j \in \{1, \cdots, p\},\ \forall\, j = 1, \cdots, n\}.$$

With simple calculation, we obtain

$$\sum_{i \in \Omega} \omega(f; I_i)|I_i| = \sum_{i=1}^{m} |I_i| = 1.$$

Hence, $f \notin \mathcal{R}([0,1]^n)$.

b) For any $\epsilon > 0$, there exists $\delta > 0$, for any partition P of I satisfying $\lambda(P) < \delta$, and for any

$\xi_i \in I_i$, we have

$$\left|\sum_{i=1}^{m} f(\xi_i)|I_i| - 0\right| = \left|\sum_{i=1}^{m}(f(\xi_i) - 0)|I_i|\right| \leq \sum_{i=1}^{m} \omega(f; I_i)|I_i| < \epsilon.$$

The last inequality holds because there always exist $y_i \in I_i$ satisfying $f(y_i) = 0$ resulting from the fact that $f(x) = 0$ holds everywhere. We complete the proof.

解答作业 4. Suppose there does not exist such interval. It follows from $f \in \mathcal{R}(I)$ that for any ϵ, there exists $\delta > 0$ so that for any partition P satisfying $\lambda(P) < \delta$, there exist $\xi_i \in I_i$ for any $i = 1, \cdots, m$ satisfying $f(\xi_i) \leq 0$ and

$$\sum_{i=1}^{m} f(\xi_i)|I_i| \leq 0,$$

which contradicts to the fact that $\int_I f(x) > 0$.

解答作业 5. Hints:

- $f(x, y) := \sin(x + y)$ is an odd function in $\Omega := [-a, a] \times [-a, a]$. Namely $(x, y) \in \Omega$, then $(-x, -y) \in \Omega$ and $f(-x, -y) = -f(x, y)$;
- $f(x, y) \in \mathcal{R}(\Omega)$;
- For any $\delta > 0$, there exists a partition P (symmetric), of Ω, satisfying $\lambda(P) < \delta$ and there exists $\xi_i \in I_i$ $(i = 1, \cdots, m)$ satisfying $\sum_{i=1}^{m} f(\xi_i)|I_i| = 0$.