中国科学院大学 2015 春季学期微积分 II-A01 习题 7

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作业 1. 计算 $\mathbb{R}^m$ 中由极坐标到笛卡尔坐标的坐标变换的 Jacobian 行列式.

作业 2. a) 设 $F: U \mapsto \mathbb{R}$ 是定义在非临界点 $x_0 = (x_0^1, \cdots, x_0^m) \in \mathbb{R}^m$ 邻域 U 上的光滑函数. 证明: 在点 x_0 的某个邻域 $\bar{U} \subset U$ 中存在曲线坐标 $(\xi^1, \cdots, \xi^m)$, 使得满足条件 $F(x) = F(x_0)$ 的 x 的点集在这个新坐标下用方程 $\xi^m = 0$ 给出.

b) 设 $\phi, \psi \in C^{(k)}(D; \mathbb{R})$, 而且, 在区域 D 中成立 $(\phi(x) = 0) \Rightarrow (\psi(x) = 0)$. 试证, 如果 $\nabla\phi = 0$, 则在 D 中成立分解式 $\psi = \theta \cdot \phi$, 这里 $\theta \in C^{(k-1)}(D; \mathbb{R})$.

作业 3. 在笛卡尔坐标 x, y 的平面 $\mathbb{R}^2$ 上定义了下面的 $C^\infty(\mathbb{R}^2, \mathbb{R})$ 类函数:

$$
\begin{aligned}
f(x,y) &= x^2 - y; \\
F(x,y) &= \begin{cases} x^2 - y + e^{-\frac{1}{x^2}} \sin \frac{1}{x}, & \text{if } x \neq 0, \\ x^2 - y, & \text{otherwise.} \end{cases}
\end{aligned}
$$

a) 画出函数 $f(x,y)$ 的等高线和由关系式 $F(x,y) = 0$ 给出的曲线 S.

b) 研究 $f|_S$ 的极值.

c) 证明: 二次型 $\partial_{ij} f(x_0)\xi^i\xi^j$ 在 TS_{x_0} 定型的条件, 与定理 2 中的二次型 $\partial_{ij} L(x_0)\xi^i\xi^j$ 在 TS_{x_0} 定型的条件不同, 并非判断函数 $f|_S$ 可疑点 $x_0 \in S$ 为极值点的充分条件.

d) 试检验, 点 $x_0 = (0,0)$ 是不是函数 f 的临界点, 以及, 能否像 c) 中那样仅借助于泰勒公式的第二项 (即二次项) 研究函数 f 在该点邻域中的性态.

作业 4. 设 $A = [a_j^i]$ 是 n 阶方阵且 $\sum\limits_{i=1}^{n} \left(a_j^i\right)^2 = H_j \quad (j = 1, \cdots, n)$, 其中 $H_1, \cdots, H_n$ 是 n 个确定的非负实数.

a) 证明: 仅当矩阵 A 的行向量是 $\mathbb{R}^n$ 中两两正交的向量时, $\det^2(A)$ 有极值.

b) 根据等式 $\det(A^2) = \det(A) \cdot \det(A^\top)$, 证明: 在上述条件下, $\max\limits_A \det^2(A) = H_1 \cdot \cdots \cdot H_n$.

c) 证明: 对任意的矩阵 $[a_j^i]$ 有阿达马不等式

$$\det{}^2 \left(a_j^i\right) \le \prod_{j=1}^{n} \left(\sum_{i=1}^{n} \left(a_j^i\right)^2 \right).$$

d) 给阿达马不等式以直观的集合解释.

当堂小测验 3

测验 1. (1 分) 证明: 若可微函数 $u=f(x,y,z)$ 满足下列方程式

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.$$

则它是一个 n 次齐次函数.[1]

测验 2. (1 分) 若

$$x^2y^2+x^2+y^2-1=0.$$

证明: 当 $xy>0$ 时, 下列等式成立

$$\frac{dx}{\sqrt{1-x^4}}+\frac{dy}{\sqrt{1-y^4}}=0.$$

测验 3. (1 分) 证明: 对于非退化的二次曲线[2]

$$ax^2+2bxy+cy^2+2dx+2ey+f=0.$$

求证:

$$\frac{d^3}{dx^3}\left[(y'')^{-\frac{2}{3}}=0\right].$$

测验 4. (2 分) 光的折射定律是

$$\frac{v_1}{v_2}=\frac{\sin\alpha_1}{\sin\alpha_2},$$

这里 $v_1(v_2)$, $\alpha_1(\alpha_2)$ 分别是光在介质 I(II) 中的传播速度和光束与介面法线的夹角. 请用“光线传播的路径是需时最少的路径”这一原理来证明光的折射定律.

[1]也即 $f(tx,ty,tz)=t^nf(x,y,z)$.

[2]也即 $\Delta:=\begin{vmatrix} a & b & d\\ b & c & e\\ d & e & f\end{vmatrix}\neq 0$ 成立.

解答作业 1. The transformation can be described as following

$$\begin{cases} x^1 = \rho\cos\phi_1, \\ x^2 = \rho\sin\phi_1\cos\phi_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x^{m-1} = \rho\sin\phi_1\sin\phi_2\cdots\sin\phi_{m-2}\cos\phi_{m-1}, \\ x^m = \rho\sin\phi_1\sin\phi_2\cdots\sin\phi_{m-2}\sin\phi_{m-1}. \end{cases}$$

The Jacobian J_m can be formulated as

$$\begin{bmatrix} \cos\phi_1 & \rho\sin\phi_1 & 0 & 0 & \cdots & 0 \\ \sin\phi_1\sin\phi_2 & \rho\cos\phi_1\cos\phi_2 & -\rho\sin\phi_1\sin\phi_2 & 0 & \cdots & 0 \\ \sin\phi_1\sin\phi_2\cos\phi_3 & \rho\cos\phi_1\sin\phi_2\cos\phi_3 & \rho\sin\phi_1\cos\phi_2\cos\phi_3 & -\rho\sin\phi_1\sin\phi_2\sin\phi_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Hence, we have

$$\det(J_m) = \rho\cos^2\phi_1\sin^{m-2}\phi_1\det(J_{m-1}) + \rho\sin^2\phi_1\sin^{m-2}\phi_1\det(J_{m-1}) = \rho\sin^{m-2}\phi_1\det(J_{m-1}).$$

Clearly $\det(J_1) = 1$, $\det(J_2) = \rho$. By mathematical induction, it holds that

$$\det(J_m) = \rho^{m-1}\prod_{i=1}^{m-1}\sin^{m-1-i}\phi_i.$$

解答作业 2. a) Since x_0 is not a stationary point, hence $\nabla F(x_0) \neq 0$. Without loss of generality, we assume $\frac{\partial F}{\partial x^m}(x_0) \neq 0$. Then we denote

$$\begin{cases} \xi^1 = x^1, \\ \xi^2 = x^2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \xi^{m-1} = x^{m-1}, \\ \xi^m = F(x^1, x^2, \cdots, x^m) - F(x_0^1, x_0^2, \cdots, x_0^m). \end{cases}$$

It is easy to verify that the Jacobian of the above transformation is invertible. Hence, we complete the proof.

b) Omitted.

解答作业 3. a) Omitted.

b) First, we study the gradient and Hessian of f

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ -1 \end{bmatrix}, \qquad \nabla^2 f(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

To introduce the expressions of the gradient and Hessian of F, we need to consider two situations.

First, if $x \neq 0$ then

$$\begin{aligned}\nabla F(x,y) &= \begin{bmatrix} 2x + \frac{1}{x^2}e^{-\frac{1}{x^2}}\left(\cos\frac{1}{x} - \frac{2}{x}\sin\frac{1}{x}\right) \\ -1 \end{bmatrix}, \\ \nabla^2 F(x,y) &= \begin{bmatrix} \left(-\frac{2}{x^3} - \frac{4}{x^5}\right)e^{-\frac{1}{x^2}}\cos\frac{1}{x} - \left(\frac{7}{x^4} - \frac{4}{x^6}\right)e^{-\frac{1}{x^2}}\sin\frac{1}{x} & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Otherwise, we have

$$\nabla F(x,y) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad \nabla^2 F(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly, $\lambda^* = 1$. Hence, besides $(0,0)$, the stationary points include all (x_0, y_0) satisfying

$$\tan\left(\frac{1}{x_0}\right) = -\frac{x_0}{2}, \quad y_0 = x_0^2 + e^{-\frac{1}{x_0^2}}\sin\frac{1}{x_0}.$$

c) Of course not. See the difference here. Note that the null space of $\nabla F(x_0, y_0)$ is $(x, 0)$.

d) Positive definiteness (negative definiteness) of $\partial_{ij}L(x_0)\xi^i\xi^j$ in the null space of $\nabla F(x_0, y_0)$ implies the local minimality (maximality), while local minimality (maximality) implies positive definiteness (negative definiteness) of $\partial_{ij}L(x_0)\xi^i\xi^j$ in the null space of $\nabla F(x_0, y_0)$.

解答作业 4. a) First we have

$$\frac{d}{dA}\det(A) = \det(A)A^{-\top}$$

. Then the first order optimality condition can be written as

$$\det(A) \cdot A^{-\top} = \Lambda A,$$

where diagonal matrix Λ contains n multipliers in its diagonal. By simple rearrangement, we have

$$\det(A) \cdot \Lambda^{-1} = AA^{\top},$$

which implies the orthogonality of among the rows of A.

b) If orthogonality holds, it holds that $\det^2(A) = \det(AA^\top) = \det(\mathrm{Diag}(H_1, \cdots, H_n)) = H_1 \cdot \cdots \cdot H_n$. In general, the determinant of a positive definite matrix is always bounded by the product of its diagonal elements (hint: Cholesky decomposition), hence, the left hand side the maximum of the right hand side.

c) Director corollary of b).

d) Suppose the lengths of the edges of a supper cube are determined, when it has maximal volume? Of course, when the edges are orthogonal to each other.