中国科学院大学 2015 春季学期微积分 II-A01 习题 6

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作业 1. 在坐标为 x, y 的平面 $\mathbb{R}^2$ 上, 用关系式 $F(x,y)=0$ (其中 $F\in C^{(2)}(\mathbb{R}^2;\mathbb{R})$) 给出了一条曲线, 设 (x_0,y_0) 位于这条曲线上且是函数 $F(x,y)$ 的非临界点.

a) 写出这条曲线在点 (x_0,y_0) 的切线方程.

b) 证明: 如果 (x_0,y_0) 是曲线的拐点, 则有等式

$$(F''_{xx}F_y^{'2}-2F''_{xy}F'_xF'_y+F''_{yy}F_x^{'2})(x_0,y_0)=0.$$

c) 求曲线在点 (x_0,y_0) 的曲率公式.

作业 2. 考察命题: 若 $f(x,y,z)=0$, 则 $\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial x}\cdot\frac{\partial x}{\partial z}=-1$.

- 给这个命题以精确的意义.
- 对于克拉贝龙定律 $\frac{P\cdot V}{T}=$ 常数, 验证上述命题的正确性, 并且对于一般的三个变量的函数验证它的正确性.
- 对于 m 个变量之间的关系式 $f(x^1,\cdots,x^m)=0$, 写出与上述类似的命题, 并验证它的正确性.

作业 3. 试证: 方程

$$z^n+c_1z^{n-1}+\cdots+c_n=0$$

的根, 当它们互不相同时光滑地依赖于方程地系数.

解答作业 1. a) Tangent equation: $F'_x(x-x_0)+F'_y(y-y_0)$.

b) We know any inflection point of $y=f(x)$ should satisfy:

$$0=f^{(2)}(x)=-\frac{[F''_{xx}+F''_{xy}f'(x)]F'_y-F'_x[F''_{xy}+F''_{yy}f'(x)]}{(F'_y)^2}. \tag{1}$$

Substituting

$$f'(x)=-F'_x/F'_y \tag{2}$$

into the above equality, we complete the proof.

c) Combining the equalities (2) and (1), the curvature at (x_0,y_0) can be formulated as

$$\kappa=\frac{|f''(x_0)|}{(1+f'^2(x_0))^{\frac{3}{2}}}=\frac{|(F''_{xx}F_y^{'2}-2F''_{xy}F'_xF'_y+F''_{yy}F_x^{'2})(x_0,y_0)|}{(F_x'^2+F_y'^2)^{\frac{3}{2}}(x_0,y_0)}.$$

解答作业 2. a) Suppose f'_x, f'_y and f'_z are all nonzero at (x_0,y_0,z_0), then according to (2), we have $\frac{\partial z}{\partial y}(x_0,y_0)\cdot\frac{\partial y}{\partial x}(x_0,z_0)\cdot\frac{\partial x}{\partial z}(y_0,z_0)=-1$

b) Let $f(P,V,T)=\frac{PV}{T}-c$, where c is a constant. Then we have $\frac{\partial T}{\partial V}=\frac{V}{T}$, $\frac{\partial V}{\partial P}=-\frac{P}{V}$, $\frac{\partial P}{\partial T}=\frac{T}{P}$ if $PV\neq 0$, which give us the result.

c)

$$\frac{\partial x^1}{\partial x^m}\cdot\prod_{i=1}^{m-1}\frac{\partial x^{i+1}}{\partial x^i}=(-1)^m.$$

解答作业 3. Let $z_1,\cdots,z_n$ to be the n distinguished roots of the polynomial $g(z):=z^n+c_1z^{n-1}+\cdots+c_n$. We define $f_i(c_1,\cdots,c_n,z_1,\cdots,z_n)=g(z_i)=0$. Let $F=(f_1,\cdots,f_n)$. Clearly

$$(F'_{z_i})_j=\begin{cases} g'(z_i), & \text{if } i=j;\\ 0, & \text{otherwise.}\end{cases}$$

Hence $F'_{(z_1,\cdots,z_n)}$ is a diagonal matrix. Its nonsingularity is equivalent to the nonzeroity of $g'(z_i)$. Suppose $g'(z_i)=0$ for some z_i, then we can prove that z_i is a multiple root of $g(z)=0$ which contradicts to the assumption. Hence, $F'_{(z_1,\cdots,z_n)}$ is nonsingular and there exists h_i so that $z_i=h_i(c_1,\cdots,c_n)$, which completes the proof.

Finally, we need to show why $g'(z_i)=0$ implies that z_i is a multiple root of $g(z)=0$. The main idea is that $g'(z)$ can be expressed by $n(z-z_i)P_{n-2}$ if z_i is its root. Then, using the technique of integration by parts and the mathematical induction, we can show $(z-z_i)^2$ is a factor of $g(z)$.