中国科学院大学 2015 春季学期微积分 II-A01 习题 5

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作业 1. 求下列函数的二阶偏导数:

(a) $f(x) = xy + \frac{x}{y}$;

(b) $f(x, y) = \ln(x^2 + y^2)$;

(c) $f(x, y, z) = e^{xyz}$.

作业 2. 设 $z = f(x, y)$ 是定义在区域 $G \subset \mathbb{R}^2$ 上的连续可微函数类 $C^{(1)}(G; \mathbb{R})$ 中的函数,

a) 如果 $\frac{\partial f}{\partial y}(x, y) \equiv 0$, $(x, y) \in G$, 那么可以断定函数 f 在 G 上不依赖于 y 吗?

b) 区域 G 具备什么条件, 才能使得上述问题有肯定的回答?

作业 3. 设 $x^1, \cdots, x^m$ 是 $\mathbb{R}^m$ 中的笛卡尔坐标, $f \in C^{(2)}(G; \mathbb{R})$. 用式子

$$\Delta f = \sum_{i=1}^{m} \frac{\partial^2 f}{\partial x^{i^2}}(x^1, \cdots, x^m)$$

定义作用在 $C^{(2)}(G; \mathbb{R})$ 上的微分算子

$$\Delta = \sum_{i=1}^{m} \frac{\partial^2 f}{\partial x^{i^2}},$$

称其为拉普拉斯算子. 在区域域 $G \subset \mathbb{R}^m$ 中关于函数 f 的方程 $\Delta f = 0$ 称为拉普拉斯方程, 它的解称为区域 G 中的调和函数.

a) 证明: 设 $x = (x^1, \cdots, x^m)$, $||x|| = \sqrt{\sum_{i=1}^{m} (x^i)^2}$, 则当 $m > 2$ 时, 函数

$$f(x) = ||x||^{2-m}$$

在区域 $\mathbb{R}^m \setminus 0$ 是调和的;

b) 证明: 函数

$$f(x, t) = \frac{1}{(2a\sqrt{\pi t})^m} e^{-\frac{||x||^2}{4a^2 t}}$$

当 $t > 0$ 且 $x = (x^1, \cdots, x^m) \in \mathbb{R}^m$ 时有定义且满足热传导方程

$$\frac{\partial f}{\partial t} = a^2 \Delta f,$$

即在函数定义域中任一点有 $\frac{\partial f}{\partial t} = a^2 \sum_{i=1}^{m} \frac{\partial^2 f}{\partial x^{i^2}}$.

作业 4. **多重指标记号的泰勒公式**记 $\alpha := (\alpha_1, \cdots, \alpha_m)$, 它由非负整数 $\alpha_i (i = 1, \cdots, m)$ 组成, 称为多重指标 α. 又记

$$\begin{aligned} |\alpha| &:= |\alpha_1| + \cdots + |\alpha_m|, \\ \alpha! &:= \alpha_1! + \cdots + \alpha_m!; \end{aligned}$$

最后, 若 $\alpha = (\alpha_1, \cdots, \alpha_m)$, 则记

$$a^\alpha = a_1^{\alpha_1} \cdot \cdots \cdot a_m^{\alpha_m}.$$

a) 证明: 若 $k \in \mathbb{N}$, 则

$$(a_1 + \cdots + a_m)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} a^\alpha.$$

b) 设

$$D^\alpha f(x) := \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha_1} \cdot \cdots \cdot (\partial x^m)^{\alpha_m}}(x).$$

证明: 若 $f \in C^{(k)}(G; \mathbb{R})$, 则对任意的点 $x \in G$, 有等式

$$\sum_{i_1 + \cdots + i_m = k} \partial_{i_1 \cdots i_m} f(x) h^{i_1} \cdot \cdots \cdot h^{i_m} = \sum_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha f(x) h^\alpha,$$

其中 $h = (h^1, \cdots, h^m)$.

c) 证明: 带有拉格朗日余项的多重指标记号的泰勒公式为

$$f(x+h) = \sum_{|\alpha|=0}^{m-1} \frac{1}{\alpha!} D^\alpha f(x) h^\alpha + \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha f(x + \theta h) h^\alpha.$$

d) 用多重指标记号写出带积分余项的泰勒公式 (定理 4).

作业 5. 将函数 $f(x, y) = x^y$ 在点 $(1, 1)$ 处作泰勒展开, 写到二次项.

作业 6. 设函数 $f(x, y)$ 在 $B = B_1(0, 0)$ 上二次连续可微而且有

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f.$$

证明: 如果在 B 的边界上 $f(x, y) \geq 0$, 则当 $(x, y) \in B$ 时, 也有 $f(x, y) \geq 0$.

作业 7. 证明下面的罗尔定理对于多变量函数的推广: 如果函数 f 在闭球 $\bar{B}(0; r)$ 上连续, 在它的边界上等于零并且在球 $B(0; r)$ 的内点可微, 则这个球至少有一个内点是函数的临界点.

作业 8. 证明: 函数 $f(x, y) = (y - x^2)(y - 3x^2)$ 在坐标原点没有极值, 虽然它在任何一条通过坐标原点的直线上的限制在原点取得局部严格极小值.

解答作业 1. (a) $\frac{\partial^2 f}{\partial x^2} = 0$; $\frac{\partial^2 f}{\partial x \partial y} = 1 - \frac{1}{y^2}$; $\frac{\partial^2 f}{\partial y^2} = \frac{2x}{y^3}$; (b) $\frac{\partial^2 f}{\partial x^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$; $\frac{\partial^2 f}{\partial x \partial y} = \frac{-4xy}{(x^2+y^2)^2}$; $\frac{\partial^2 f}{\partial y^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$. (c) $\frac{\partial^2 f}{\partial x^2} = y^2z^2e^{xyz}$ $\frac{\partial^2 f}{\partial x \partial y} = z(1+xyz)e^{xyz}$; $\frac{\partial^2 f}{\partial x \partial z} = y(1+xyz)e^{xyz}$; $\frac{\partial^2 f}{\partial y^2} = x^2z^2e^{xyz}$; $\frac{\partial^2 f}{\partial y \partial z} = x(1+xyz)e^{xyz}$; $\frac{\partial^2 f}{\partial z^2} = x^2y^2e^{xyz}$.

解答作业 2. a) No, for instance,

$$f(x,y) = \begin{cases} x^2, & \text{if}\, x > 0 \,\text{and}\, y > 0; \\ 0, & \text{otherwise.} \end{cases}$$

$G := \{(x,y) \mid (x \geq 0) \wedge (y = 0) = 0\}$; b) The region that f restricted to some fixed x is a connected interval of y.

解答作业 3. Using the fact that $||x||' = x/||x||$.

解答作业 4. Omitted.

解答作业 5.

$$\begin{aligned} f(x,y) &= f(1,1) + \nabla f(1,1)^\top \begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}^\top \nabla^2 f(1,1) \begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + o\left(\left\| \begin{matrix} x-1 \\ y-1 \end{matrix} \right\|^2 \right) \\ &= 1 + \begin{matrix} 0 & x-1 \\ 1 & y-1 \end{matrix} + \frac{1}{2} \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}^\top \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + o\left(\left\| \begin{matrix} x-1 \\ y-1 \end{matrix} \right\|^2 \right) \\ &= 1 + (x-1) + (x-1)(y-1) + o\left(\left\| \begin{matrix} x-1 \\ y-1 \end{matrix} \right\|^2 \right) = 1 - y + xy + o\left(\left\| \begin{matrix} x-1 \\ y-1 \end{matrix} \right\|^2 \right). \end{aligned}$$

解答作业 6. Suppose that there exists $(x', y') \in B_1(0,0)$ such that $f(x,y) < 0$. Then if we denote $(x^*, y^*) = \underset{(x,y \in B_1(0,0))}{\arg\min} \; f(x,y)$, we have $(x^*, y^*) \in B_1(0,0) \setminus \partial B_1(0,0)$ and $f(x^*, y^*) < 0$. On the other hand, due to the optimality, it holds that $\nabla f(x^*, y^*) = 0$ and $\nabla^2 f(x^*, y^*) \geq 0$ which implies that $\frac{\partial^2 f}{\partial x^2}(x^*, y^*) + \frac{\partial^2 f}{\partial y^2}(x^*, y^*) \geq 0$ which contradicts to $f(x^*, y^*) < 0$.

解答作业 7. First, let $x_1 = \underset{x \in \bar{B}(0;r)}{\arg\min} f$, $x_2 = \underset{x \in \bar{B}(0;r)}{\arg\max} f$. If $x_1 = x_2$, then f is constant in $\bar{B}(0;r)$ and consequently each point is stationary point. Otherwise, at least one of x_1 or x_2 is not in $\partial \bar{B}(0;r)$. Without loss of generality, we assume $x_1 \notin \partial \bar{B}(0;r)$, which implies that $\nabla f(x_1) = 0$.

解答作业 8. First, it clear that f increases along $x = 0$ at $(0,0)$, and f decrease along $y = 2x^2$ at $(0,0)$. Thus, $(0,0)$ is neither minimizer nor maximizer. Then consider $x = 0$ and $y = tx$. For the first case, the statement holds obviously. For the second case, $g(x) = f(x, tx)$, it is also easy to prove the result.