中国科学院大学 2014 秋季学期微积分 II-A01 习题 4

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作业 1. 计算偏导数:

(a) $f(x) = xy + \frac{x}{y}$;

(b) $f(x, y, z) = e^{xyz}$;

(c) $f(x, y) = \ln(x + y^2)$.

作业 2. 设 X 是 $\mathbb{R}^2$ 中的凸区域, f 的两个偏导数在 X 上有界, 证明 f 在 X 上一致连续.

作业 3. 求证:

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{otherwise.} \end{cases}$$

在 $(0,0)$ 可微, 但偏导数在 $(0,0)$ 不连续.

作业 4. 函数值计算的误差估计.

a) 利用可微函数定义与近似等式 $\Delta f(x; h) \approx df(x)h$, 证明: 设 $f(x) = x^1 \cdot \cdots \cdot x^m$ 是 m 个不为零的因子的乘积, 若 δ_i 是第 i 个因子的相对误差, 则它们乘积的相对误差为 $\delta = \delta(f(x); h) \approx \sum\limits_{i=1}^{m} \delta_i$.

b) 利用等式 $d \ln f(x) = \frac{1}{f(x)} df(x)$, 再次推导上题结果, 并证明: 一般的分式

$$\frac{f_1 \cdot \cdots \cdot f_n}{g_1 \cdot \cdots \cdot g_k}(x^1, \cdots, x^m)$$

的相对误差是函数 $f_1, \cdots, f_n, g_1, \cdots, g_k$ 的值的误差的和.

作业 5. a) 画出函数 $z = x^2 + 4y^2$ 的图像, 其中 x, y, z 是 $\mathbb{R}^3$ 空间中的笛卡尔坐标.

b) 设 $f: G \to \mathbb{R}$ 是定义在 $G \subset \mathbb{R}^m$ 上的数值函数, 如果函数 f 在集合 $E \subset G$ 上仅有一个值 $(f(E) = c)$, 精确些说, $E = f^{-1}(c)$, 则称集合 E 是函数 f 的等高集 (c-等高面). 在 $\mathbb{R}^2$ 中画出函数 $f(x, y) = x^2 + 4y^2$ 的等高集的图像.

c) 求函数 $f(x, y) = x^2 + 4y^2$ 的梯度, 并且证明: 在任一点 (x, y) 处, 向量 ∇f 与 f 过此点的等高线垂直.

d) 利用上述 a), b), c) 的结果, 在曲面 $z = x^2 + 4y^2$ 上求出点 $(2, 1, 8)$ 到它的最低点 $(0, 0, 0)$ 的最短路径.

e) 为了寻找数 $f(x, y) = x^2 + 4y^2$ 的最小值, 你能提出什么适用于电子计算机的算法?

解答作业 1. (a) $\frac{\partial f}{\partial x} = y + \frac{1}{y}$; $\frac{\partial f}{\partial y} = x - \frac{x}{y^2}$; (b) $\frac{\partial f}{\partial x} = yze^{xyz}$; $\frac{\partial f}{\partial y} = xze^{xyz}$; $\frac{\partial f}{\partial z} = xye^{xyz}$; (c) $\frac{\partial f}{\partial x} = \frac{1}{x+y^2}$; $\frac{\partial f}{\partial y} = \frac{2y}{x+y^2}$.

解答作业 2. Since the partial derivatives are bounded in X, there exists $M > 0$ such that $|\partial_1 f(x,y)| < M$ and $|\partial_2 f(x,y)| < M$ hold for any $(x,y) \in X$. For any $\epsilon > 0$, we set $\delta = \epsilon/2M$. For any (x_1,y_1), $(x_2,y_2) \in X$, we have $(x_1 + \theta(x_2 - x_1), y_1 + \theta(y_2 - y_1)) \in X$ due to the convexity, where $\theta \in (0,1)$. Hence, we have

$$
\begin{aligned}
& |f(x_1,y_1) - f(x_2,y_2)| \\
= \; & |f(x_1,y_1) - f(x_1+\theta(x_2-x_1), y_1+\theta(y_2-y_1)) + f(x_1+\theta(x_2-x_1), y_1+\theta(y_2-y_1)) - f(x_2,y_2)| \\
\leq \; & |f(x_1,y_1) - f(x_1+\theta(x_2-x_1), y_1+\theta(y_2-y_1))| + |f(x_1+\theta(x_2-x_1), y_1+\theta(y_2-y_1)) - f(x_2,y_2)| \\
= \; & |f'_\theta(f(x_1+\theta(x_2-x_1), y_1+\theta(y_2-y_1)))\theta| + o(||\theta(x_2-x_1)|| + ||\theta(y_2-y_1)||) \\
& |f'_\theta(f(x_1+\theta(x_2-x_1), y_1+\theta(y_2-y_1)))(1-\theta)| + o(||(1-\theta)(x_2-x_1)|| + ||(1-\theta)(y_2-y_1)||) \\
= \; & |\partial_1 f(f(x_1+\theta(x_2-x_1), y_1+\theta(y_2-y_1)))(x_2-x_1) \\
& +\partial_2 f(f(x_1+\theta(x_2-x_1), y_1+\theta(y_2-y_1)))(y_2-y_1)| + o(||x_2-x_1|| + ||y_2-y_1||) \\
\leq \; & \epsilon.
\end{aligned}
$$

We complete the proof.

解答作业 3. We just study the case of $\partial_1 f(x,0)$. For $x \neq 0$, we have $\partial_1 f(x,0) = 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}$, when x approaches to zero, $\partial_1 f(x,0)$ does not converge. For $x \neq 0$, we have $\partial_1 f(0,0) = \lim\limits_{x\to 0} x \sin\frac{1}{x^2+y^2} = 0$. We complete the proof.

解答作业 4. a) $\delta \cdot f(x) = f(x_1 + \Delta x_1, \cdots, x_m + \Delta x_m) - f(x_m, \cdots, x_m) = \prod\limits_{i=1}^m (x_i + \Delta x_i) - \prod\limits_{i=1}^m x_i = \prod\limits_{i=1}^m x_i(1+\delta_i) - \prod\limits_{i=1}^m x_i = \left(\sum\limits_{i=1}^m \delta_i\right) \cdot \prod\limits_{i=1}^m x_i + o(\delta_i)$.

b) $\delta = \frac{df(x)h}{f(x)} = d\ln f(x)h = \sum\limits_{i=1}^m \frac{1}{x_i}((1+\delta_i)x_i - x_i) = \sum\limits_{i=1}^m \delta_i$. The general form can be derived in the same manner, this is because

$$
d\ln f(x) = \frac{f'(x)dx}{f(x)} = \frac{df(x)}{f(x)} = \delta_f,
$$

where δ_f refers to the relative error of $f(x)$.

解答作业 5. c) $\nabla f(x,y) = \begin{pmatrix} 2x \\ 8y \end{pmatrix}$. This is to prove $\left\langle \begin{pmatrix} 2x \\ 8y \end{pmatrix}, \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \right\rangle = 0$ when $(\Delta x, \Delta y) \to (0,0)$, where $f(x+\Delta x, y+\Delta y) = f(x,y)$.

d) The shortest path should follow the gradient flow.

e) Steepest descent method.

1. Set desired precision. Set initial guess: (x_0, y_0) and $k = 0$;
2. Calculate the gradient $g_k = \nabla f(x_k, y_k)$;
3. If $||\nabla f(x_k, y_k)|| < \epsilon$, stop; otherwise goto step 4.

4. Compute the step size: $\alpha_k = \arg\min f(x_k - \alpha g_k^x, y_k - \alpha g_k^y)$;
5. Update iterate: $(x_{k+1}, y_{k+1}) = (x_k, y_k) - \alpha_k g_k$;
6. Set $k := k + 1$ and goto step 2.