中国科学院大学 2014 秋季学期微积分 II-A01 习题 3

课程教师：袁亚湘　　助教：刘歆

2015 年 3 月 22 日，8:00-9:40

作业 1. 证明:

a) 连通集 E 在连续映射 $f: E \to \mathbb{R}^n$ 下的像 $f(E)$ 是连通集.

b) 有公共点的连通集的并是连通集.

c) 半球面 $x_1^2 + \cdots x_n^2 = 1$, $x_n \geq 0$, 是连通集.

d) 球面 $x_1^2 + \cdots x_n^2 = 1$ 是连通集.

e) 若 $E \subset \mathbb{R}$ 是连通集, 则 E 是 $\mathbb{R}$ 上的区间.

f) 若 x_0 与 x_1 分别是集合 $M \subset \mathbb{R}^n$ 的内点和外点, 则以 x_0 与 x_1 为端点的任何一条通路的承载子与集合 M 的边界相交.

作业 2. 设二元函数 $f(x,y)$ 在 (x_0,y_0) 处的某一去心邻域上有定义. 对任意固定的 y, 如果极限 $\lim\limits_{x\to x_0} f(x,y)$ 存在, 将其记为 $\phi(y)$. 如果 $\lim\limits_{y\to y_0} \phi(y)$ 存在, 我们就称其为 $f(x,y)$ 在 (x_0,y_0) 的一个累次极限, 记为

$$\lim_{y\to y_0}\lim_{x\to x_0} f(x,y) = \lim_{y\to y_0} (\phi(y)).$$

类似地, 我们可以定义

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y) = \lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right).$$

请计算 $f(x,y) = \frac{x^2 y}{x^4+y^2}$ 在 $(0,0)$ 处的两个累次极限.

作业 3. 设 $f(x,y) = (x,y)\sin\frac{1}{x}\sin\frac{1}{y}$ 在 $(0,0)$ 的两个累次极限都不存在, 但极限 $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ 存在.

作业 4. 证明: 如果二元函数在某点的极限以及两个累次极限都存在, 则这三个极限必相等.

作业 5. 证明: 线性映射是一致连续的.

作业 6. 设 $f: \mathbb{R}^n \to \mathbb{R}^n$ 是线性映射, 则存在矩阵 $A \in \mathbb{R}^{n\times n}$ 使得 $f(x) = Ax$. 线性映射 f 存在可逆映射的充分必要条件是矩阵 A 是可逆矩阵.

作业 7. 设 $X \subset \mathbb{R}^n$ 是紧集, $f: X \to \mathbb{R}^n$ 是 X 上的连续单射, 记 $f(X) = Y \subset \mathbb{R}^n$. 证明映射 f^{-1} 在 Y 上连续.

作业 8. 证明: 不存在从 $[0,1]$ 到单位圆周上的一对一的连续映射.

当堂小测验 2

测验 1. 设 $f(x,y)$ 定义在开集 Ω 内, 若 $f(x,y)$ 对 x 连续, 对 y 满足如下 Lipschitz 条件: 存在常数 $L>0$, 使得对任意 (x,y'), $(x,y")\in\Omega$, 有

$$|f(x,y')-f(x,y")|\leq L|y'-y"|.$$

求证: $f(x,y)$ 在 Ω 上连续.

测验 2. 设 $f(x)$ 在 $\mathbb{R}^n$ 上连续, 满足:

(1) $x\neq 0$ 时, $f(x)>0$;

(2) 对任意 x 和正常数 c, 有 $f(cx)=cf(x)$ 成立.

求证: 存在 $a>0$, $b>0$, 使得 $a|x|\leq f(x)\leq b|x|$.

解答作业 1. a) Using the transitivity of continuity, we complete the proof. b) If the two points to be connected are not in the same connected set, we construct a path passing one point in the intersection part then. c) x, y, then $(\lambda x+(1-\lambda)y)/||\lambda x+(1-\lambda)y||$ is the path. d) Corollary of b) and c). e) Obviously. f) Let $P(t)$ be the path, $x_0 = P(0)$, $x_1 = P(1)$. Set $t^* = \sup\limits_{t\in[0,1]}\{t \mid P(s)\in M, \quad \forall\, s\in[0,t]\}$. We can prove that $P(t^*)\in\partial M$.

解答作业 2. It is easily to obtain $\phi(y) = \begin{cases} 0 & \text{if}\, y\neq 0; \\ 0 & \text{otherwise}, \end{cases}$ and hence $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} = 0$. Similarly, $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} = 0$.

解答作业 3. For any fixed $y\neq 0$, there exists $\epsilon = y$, for any $\delta > 0$, the amplitude of function $f(x,y)$ with respect to x on the set $B^o_\delta(0)$ is equal to $2y$ which is great than ϵ. Hence the repeated limit $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$ at $(0,0)$ does not exist. Similarly, the repeated limit $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y)$ at $(0,0)$ does not exist either.

On the other hand, for any $\epsilon > 0$, we set $\delta = \epsilon/\sqrt{2}$. Then we have $|f(x,y)-0|\le|x+y|\le\epsilon$. Namely, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

解答作业 4. Let $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y) = a$, $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = b$. Suppose $a\neq b$. Set $\epsilon = \frac{|a-b|}{3}$, there exists $\delta > 0$ such that $|f(x,y)-b| < \epsilon$ for any $(x,y)\in B_\delta(x_0,y_0)$. On the other hand, there exists $\delta' \le \frac{\delta}{\sqrt{2}}$ such that $|\phi(y)-a| < \frac{\epsilon}{2}$, for any $y\in B_{\delta'}(y_0)$; there exists $\delta'_y \le \frac{\delta}{\sqrt{2}}$ such that $|f(x,y)-\phi(y)| < \frac{\epsilon}{2}$, for any $x\in B_{\delta'_y}(x_0)$. Namely, $|f(x,y)-a| < \epsilon$ for any $(x,y)\in\Omega := \{(x,y)\mid y\in B_{\delta'}(y_0), x\in B_{\delta'_y}(x_0)\}$, clearly $\Omega\subset B_\delta(x_0,y_0)$, and hence $|b-a|\le|f(x,y)-b|+|f(x,y)-a| < \frac{2|a-b|}{3}$, which lead to contradiction. Hence, $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$. We can prove $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ in the same manner. We complete the proof.

解答作业 5. For any $\epsilon > 0$, let $\delta = \frac{\epsilon}{||A||}$, then $||Ax-Ay|| = ||A(x-y)||\le||A||\cdot||x-y|| < \epsilon$ holds for any x,y satisfying $||x-y|| < \delta$.

解答作业 6. First, if A is invertible, we set $g(x) = A^{-1}x$, and can prove $g = f^{-1}$ is the inverse mapping of f. Secondly, if the inverse mapping of f exists and A is not invertible, there exists $y\neq 0$ such that $Ay = 0$, hence $f^{-1}(y) = f^{-1}(0)$ which is contrary to the fact that f^{-1} is an injection. Hence, A is invertible. We complete the proof.

解答作业 7. Suppose f^{-1} is not continuous at y_0, which means there exists $\epsilon > 0$, for any $n\in\mathbb{N}$, there exists y_n satisfying $|y_n-y_0| < \frac{1}{n}$, such that $|f^{-1}(y_n)-f^{-1}(y_0)| > \epsilon$. Since $\{f^{-1}(y_n)\}\subset X$, hence there exists a subsequence $\{f^{-1}(y_{j_n})\}$ such that $\lim\limits_{n\to\infty} f^{-1}(y_{j_n}) = \bar{x}$. Hence, we have

(i) $\lim\limits_{n\to\infty} y_{j_n} = f(\bar{x})$;

(ii) $\lim\limits_{n\to\infty} y_{j_n} = y_0$;

Combing (i)-(ii), we obtain that $f(\bar{x}) = y_0$, and then $\bar{x} = f^{-1}(y_0)$ due to the bijection of f^{-1}, which contradicts to $|f^{-1}(y_{j_n})-f^{-1}(y_0)| > \epsilon$ and $\lim\limits_{n\to\infty} f^{-1}(y_{j_n}) = \bar{x}$. We complete the proof.

解答作业 8. Suppose there exists $f: [0,1]\mapsto B_1(0,0)$. According to injection, $f(0)$ and $f(1)$ are two different points on the unit circumference. We let $X\subset(0,1)$ to be $f^{-1}(A_1)$, and then $(0,1)\setminus X =$

$f^{-1}(A_2)$, where A_1 and A_2 are the two open arcs from $f(0)$ to $f(1)$. According to Theorem 9.7.9, X is an open set. Then it follows from Homework 8 in the last assignment, we have X is the union of countable number of open interval. Thus, $(0,1) \setminus X$ is not an open set, and which is contrary to Theorem 9.7.9, since A_2 is also open.