中国科学院大学 2015 春季学期微积分 II-A01 习题 15

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作业 1. 计算

$$\int_L ydx + zdy + xdz$$

其中 L 是平面 $x+y=2$ 和球面 $x^2+y^2+z^2=2(x+y)$ 交成的圆周, 从原点看去, 顺时针方向是曲线 L 的积分方向.

作业 2 (P225-4). a) 试证, 如果格林公式中的函数 P, Q 满足 $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, 则区域 D 的面积 $\sigma(D)$, 可用公式 $\sigma(D) = \int_{\partial D} Pdx + Qdy$ 求得.

b) 设 x, y 是平面上的笛卡尔坐标, γ 是平面上一条曲线 (可能不闭). 说明积分 $\int_\gamma ydx$ 的几何意义. 由此出发, 重新解释公式

$$\sigma(D) = -\int_{\partial D} ydx.$$

c) 利用这个公式求区域 $D = \left\{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}$ 的面积, 以验证 b) 中的公式

作业 3 (P225-5). a) 设 $x = x(t)$ 是从区域 $D_t \subset \mathbb{R}^2_t$ 上的微分同胚. 利用问题 4 的结果, 以及曲线积分关于路径的容许参数变换之无关性, 试证:

$$\int_{D_x} dx = \int_{D_t} |x'(t)| dt,$$

其中 $dx = dx^1dx^2$, $dt = dt^1dt^2$, $|x'(t)| = \det x'(t)$.

b) 由 a) 导出二重积分中的变量替换公式

$$\int_{D_x} f(x)dx = \int_{D_t} f(x(t)) |\det x'(t)| dt.$$

作业 4 (P245-1). 算子 grad, rot, div 及代数运算. 验证以下各式, 并用符合 grad, rot, div 把他们表出: 关于 grad:

a) $\nabla(f+g) = \nabla f + \nabla g$,

b) $\nabla(f \cdot g) = f\nabla g + g\nabla f$,

c) $\nabla(A \cdot B) = (B \cdot \nabla)A + (A \cdot \nabla)B + B \times (\nabla \times A) + A \times (\nabla \times B)$,

d) $\nabla\left(\frac{1}{2}A^2\right) = (A \cdot \nabla)A + A \times (\nabla \times A)$.

关于 rot

e) $\nabla \times (fA) = f\nabla \times A + \nabla f \times A$,

f) $\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B + (\nabla \cdot B)A - (\nabla \cdot A)B$.

关于 div

g) $\nabla \cdot (fA) = \nabla f \cdot A + f\nabla \cdot A$,

h) $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$.

作业 5 (P245-2). a) 将 (20)-(22) 内的算子用笛卡尔坐标写出.

b) 用直接计算的方法验证 (20) 式及 (21) 式.

c) 验证表示成笛卡尔坐标形式的公式 (24).

d) 用算子 ∇ 写出公式 (24), 并用向量代数式验证它.

解答作业 1.

$$\int_L ydx + zdy + xdz = -\iint_D (dy\wedge dz + dz\wedge dx + dx\wedge dy)$$

这里 $D := \{(x,y) \mid x+y=2, \quad (x-1)^2+(y-1)^2+z^2 \le 2\}$, 因此参数方程为

$$\begin{cases} x &= 1+r\sin\theta \\ y &= 1+r\sin\theta \\ z &= \sqrt{2}\cos\theta \end{cases} \qquad (r,\theta)\in I := \{0\le r\le 1,\, 0\le\theta\le 2\pi\}$$

因此 $\frac{\partial x}{\partial\theta} = r\cos\theta$, $\frac{\partial y}{\partial\theta} = -r\cos\theta$, $\frac{\partial z}{\partial\theta} = -\sqrt{2}r\sin\theta$, $\frac{\partial x}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial r} = -\sin\theta$, $\frac{\partial z}{\partial r} = \sqrt{2}\cos\theta$.

$$\begin{aligned} F &= \iint_D dy\wedge dz + dz\wedge dx + dx\wedge dy \\ &= \int_I (-\sqrt{2}r\cos^2\theta - \sqrt{2}r\sin^2\theta - \sqrt{2}r\cos^2\theta - \sqrt{2}r\sin^2\theta - r\sin\theta\cos\theta + r\sin\theta\cos\theta)d\theta dr \\ &= -2\sqrt{2}\int_I rd\theta dr = -2\sqrt{2}\cdot 2\pi\int_0^1 rdr = -2\sqrt{2}\pi. \end{aligned}$$

又因为定向为顺时针, $\int_L ydx+zdy+xdz = F = -2\sqrt{2}\pi$.

解答作业 2. a)

$$\sigma(D) = \int_D 1\cdot dxdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial D} Qdy + Pdx.$$

b) 记 γ 的起点和终点分别为 a,b, 若 γ 闭合, 则 $a=b$.

$$\int_\gamma ydx + \int_{l_1} ydx + \int_{l_2} ydx = \int_{\partial D} ydx = -\iint_D dxdy.$$

$\int_{l_2} ydx = 0$, $\int_{l_1} ydx = y_a(x_b - x_a)$. 因此 $\int_\gamma ydx = -\iint_D dxdy - y_a(x_b-x_a)$. 因此 $\int_y dx$ 的几何意思为曲线 γ 定义函数的积分的负数. 特别的, 当 γ 闭合时, $\int_{l_1} ydx = y_a(x_b-x_a) = 0$, $\sigma(D) = -\int_\gamma ydx = -\int_{\partial D} ydx$.

c) 令 $x = a\cos\theta$, $y = b\sin\theta$, 则

$$\sigma(D) = -\int_0^{2\pi} -b\sin\theta a\sin\theta d\theta = ab\int_0^{2\pi}\frac{1-\cos\theta}{2}d\theta - \pi ab.$$

解答作业 3. a) 令 $Q = x^1$, $P=0$, 则有

$$\int_{D_x} dx = \int_{\partial D_x} x^1 dx^2.$$

更进一步, 我们有

$$\int_{\partial D_x} x^1 dx^2 \quad = \quad \int_{\partial D_x} x^1(t^1,t^2)\left(\frac{\partial x^2}{\partial t^1}dt^1 + \frac{\partial x^2}{\partial t^2}dt^2\right)$$

$$
\begin{aligned}
&= \int_{D_t} -\frac{\partial x^1}{\partial t^2}\frac{\partial x^2}{\partial t^1}dt^1dt^2 + \frac{\partial x^1}{\partial t^1}\frac{\partial x^2}{\partial t^2}dt^1dt^2 \\
&= \int_{D_t} |x'(t)|dt.
\end{aligned}
$$

b) 类似可证, 略.

解答作业 4. a) 可表达为 $\mathrm{grad}(f\cdot g) = f\cdot \mathrm{grad}g + g\cdot\mathrm{grad}f$,

$$
\mathrm{grad}(f+g) = e_1\frac{\partial f+g}{\partial x^1} + e_2\frac{\partial f+g}{\partial x^2} + e_3\frac{\partial f+g}{\partial x^3} = \mathrm{grad}f + \mathrm{grad}g.
$$

b) 可表达为 $\mathrm{grad}(f\cdot g) = f\cdot\mathrm{grad}g + g\cdot\mathrm{grad}f$, 证明略.

c) 可表达为 $\mathrm{grad}(A\cdot B) = (B\cdot\mathrm{grad})A + (A\cdot\mathrm{grad})B + B\times\mathrm{rot}A + A\times\mathrm{rot}B$.

$$
\begin{aligned}
\mathrm{grad}(A\cdot B) &= \sum_{1\le i,j\le 3} e_i\frac{\partial A^jB^j}{\partial x^i}, \\
B\times(\mathrm{grad}\times A) &= B\times\left(e_1\left(\frac{\partial A^3}{\partial x^2}-\frac{\partial A^2}{\partial x^3}\right)+e_2\left(\frac{\partial A^1}{\partial x^3}-\frac{\partial A^3}{\partial x^1}\right)+e_3\left(\frac{\partial A^1}{\partial x^3}-\frac{\partial A^3}{\partial x^1}\right)\right), \\
(B\cdot\nabla)A &= \left(B^1\frac{\partial}{\partial x^1}+B^2\frac{\partial}{\partial x^2}+B^3\frac{\partial}{\partial x^3}\right)(A^1e^1+A^2e^2+A^3e^3).
\end{aligned}
$$

整理后证毕.

d) 令 c) 中 $B = A$, 可得.

e) 可表达为 $\mathrm{rot}(fA) = f\cdot\mathrm{rot}A + \mathrm{grad}f\times A$.

f) 可表达为 $\mathrm{rot}(A\times B) = (B\cdot\nabla)A - (A\cdot\nabla)B + (\mathrm{div}B)\cdot A - (\mathrm{div}A)\cdot B$.

g) 可表达为 $\mathrm{div}(fA) = (\mathrm{grad}f)\cdot A + f\cdot\mathrm{div}A$.

h) 可表达为 $\mathrm{div}(A\times B) = B\mathrm{rot}A - A\mathrm{rot}B$.

解答作业 5. a)

$$
\mathrm{rotgrad}f = \mathrm{rot}\left(\frac{\partial f}{\partial x^1}e_1+\frac{\partial f}{\partial x^2}e_2+\frac{\partial f}{\partial x^3}e_3\right)
$$

$$
= e_1\left(\frac{\partial^2 f}{\partial x^2\partial x^3}-\frac{\partial^2 f}{\partial x^3\partial x^2}\right)+e_2\left(\frac{\partial^2 f}{\partial x^3\partial x^1}-\frac{\partial^2 f}{\partial x^1\partial x^3}\right)+e_3\left(\frac{\partial^2 f}{\partial x^1\partial x^2}-\frac{\partial^2 f}{\partial x^2\partial x^1}\right)=0
$$

$$
\mathrm{divrot}f = \mathrm{div}\left(e_1\left(\frac{\partial A^3}{\partial x^2}-\frac{\partial A^2}{\partial x^3}\right)+e_2\left(\frac{\partial A^1}{\partial x^3}-\frac{\partial A^3}{\partial x^1}\right)+e_3\left(\frac{\partial A^2}{\partial x^1}-\frac{\partial A^1}{\partial x^2}\right)\right)
$$

$$
= \frac{\partial}{\partial x^1}\left(\frac{\partial A^3}{\partial x^2}-\frac{\partial A^2}{\partial x^3}\right)+\frac{\partial}{\partial x^2}\left(\frac{\partial A^1}{\partial x^3}-\frac{\partial A^3}{\partial x^1}\right)+\frac{\partial}{\partial x^3}\left(\frac{\partial A^2}{\partial x^1}-\frac{\partial A^1}{\partial x^2}\right)=0.
$$

$$
\begin{aligned}
\mathrm{graddiv}A &= \sum_{1\le i,j\le 3} e_i\left(\frac{\partial^2 A^j}{\partial x^i x^j}\right), \\
\mathrm{rotrot}A &= \sum_{i=1}^{3} e_i\sum_{j\ne i}\left(\frac{\partial^2 A^j}{\partial x^i x^j}-\frac{\partial^2 A^i}{\partial x^j x^j}\right),
\end{aligned}
$$

$$\mathrm{divgrad}f \quad = \quad \mathrm{div}\left(e_1\frac{\partial f}{\partial x^1}+e_2\frac{\partial f}{\partial x^2}+e_3\frac{\partial f}{\partial x^3}\right)=\Delta f.$$

b) 即 a). c) 由 a) 易得. d) 公式

$$\mathrm{rotrot}A=\mathrm{graddiv}A-\Delta A$$

可写为

$$\nabla\times(\nabla\times A)=\nabla(\nabla\cdot A)-\nabla^2 A.$$

用 $A\times(B\times C)=B(A\cdot C)-C(A\cdot B)$ 立即可得.