中国科学院大学 2015 春季学期微积分 II-A01 习题 11

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作业 1. 计算 $\mathbb{R}^3$ 中椭球 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (其中 $a,b,c>0$) 的体积.

作业 2. 令 $D=\{(x_1,\cdots,x_n)\in\mathbb{R}^n \mid x_1^2+x_2^2+\cdots x_n^2\le\mathbb{R}^2\}$. 将重积分

$$\int\int\cdots\int_D f\left(\sqrt{x_1^2+x_2^2+\cdots+x_n^2}\right)dx_1dx_2\cdots dx_n$$

化为单变量积分.

作业 3. 萨德 (Sard) 引理, 设 D 是 $\mathbb{R}^n$ 中的开集, $\varphi\in C^{(1)}(D,\mathbb{R}^n)$, S 是映射 φ 的临界点集, 那么 $\varphi(S)$ 是勒贝格零测度集合. (我们提醒大家, 区域 $D\subset\mathbb{R}^m$ 到空间 $\mathbb{R}^n$ 中的光滑映射 φ 的临界点是指使 $\varphi'(x)$ 的秩小于 $\min\{m,n\}$ 的点 $x\in D$. 当 $m=n$ 时, 这相当于条件 $\det\varphi'(x)=0$.) 对线性映射验证萨德引理.

作业 4. 指出对于怎样的参数 p,q, 积分 $\iint\limits_{0<|x|+|y|\le1}\frac{dxdy}{|x|^p+|y|^q}$ 收敛.

作业 5. a) $\lim\limits_{A\to\infty}\int_0^A\cos x^2dx$ 是否存在?

b) 积分 $\int_{\mathbb{R}^1}\cos^2 xdx$ 在定义 2 的意义下是否收敛?

c) 验证

$$\lim_{n\to\infty}\iint\limits_{|x|\le n}\sin(x^2+y^2)dxdy=\pi$$

和

$$\lim_{n\to\infty}\iint\limits_{x^2+y^2\le2n\pi}\sin(x^2+y^2)dxdy=0,$$

证明 $\sin(x^2+y^2)$ 在平面 $\mathbb{R}^2$ 上的积分发散.

作业 6. 若 $f\in C(\mathbb{R},\mathbb{R})$, 则

$$\lim_{h\to0}\frac{1}{\pi}\int_{-1}^{1}\frac{h}{h^2+x^2}f(x)dx=f(0).$$

作业 7. 证明: 对于 $\mathbb{R}^2$ 中的函数 $f(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2}$ 虽然累次积分

$$\int_1^{+\infty}dx\int_1^{+\infty}f(x,y)dy,\qquad\int_1^{+\infty}dy\int_1^{+\infty}f(x,y)dx$$

都存在. 但反常积分

$$\iint\limits_{x\geq 1, y\geq 1} f(x,y)dxdy$$

不存在.

当堂小测验 6

测验 1. 计算积分

$$I = \iint\limits_{x^2+y^2\leq 1} |3x+4y|dxdy.$$

测验 2. 计算重积分

$$I = \iint\limits_{D} xdxdy,$$

其中 D 是以 $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ 为顶点的三角形.

解答作业 1.

$$\begin{aligned}
V &= \int_{-c}^{c} dz \int_{-b\sqrt{1-\frac{z^2}{c^2}}}^{b\sqrt{1-\frac{z^2}{c^2}}} dy \int_{-a\sqrt{1-\frac{z^2}{c^2}-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{z^2}{c^2}-\frac{y^2}{b^2}}} dx \\
&= \int_{-c}^{c} dz \int_{-b\sqrt{1-\frac{z^2}{c^2}}}^{b\sqrt{1-\frac{z^2}{c^2}}} 2a\sqrt{1-\frac{z^2}{c^2}-\frac{y^2}{b^2}}dy \\
&= \int_{-c}^{c} \pi ab\left(1-\frac{z^2}{c^2}\right) dz = \frac{4\pi}{3}abc.
\end{aligned}$$

解答作业 2. Let $I := \{(\rho, \phi_1, \cdots, \phi_{n-1}) \mid 0 \le \rho \le R, 0 \le \phi_i \le \pi, \forall i = 1, \cdots, n-1\}$. Here, of course, we need to assume that $n \ge 2$. $\varphi : I \setminus \partial I \mapsto D$. Then according to Theorem 12.5.13, we have

$$\int\int \cdots \int_D f\left(\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\right) dx_1 dx_2 \cdots dx_n = \int_I f(\rho)|\det(\varphi')| d\rho d\phi_1 \cdots d\phi_{n-1}. \tag{1}$$

According to Homework 7/1, we have

$$\det(\varphi') = \rho^{n-1} \prod_{i=1}^{n-1} \sin^{n-1+i} \phi_i. \tag{2}$$

Substitute (2) into (1), we obtain

$$\int\int \cdots \int_D f\left(\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\right) dx_1 dx_2 \cdots dx_n = c \cdot \int_0^R f(\rho)\rho^{n-1} d\rho, \tag{3}$$

where

$$c = \begin{cases} \frac{(2\pi)^{n/2+1} \cdot 2^n}{(n-2)!!} & \text{if } n \text{ is even}, \\ \frac{(8\pi)^{(n-1)/2} \cdot 2^n}{(n-2)!!} & \text{otherwise}. \end{cases}$$

解答作业 3. Since φ is linear, hence $\varphi'(x)$ is a constant. (1) $\varphi'(x) \neq 0$ $(\forall\, x \in D)$, then $S = \emptyset$ which directly implies $\varphi(S)$ is a zero measurement set. (2) $\varphi'(x) = 0$ $(\forall\, x \in D)$, then, according to Theorem 12.5.13

$$\mu(\varphi(D)) = \int_{\varphi(D)} 1 dt = \int_D 1 |\det \varphi'(x)| dx = 0.$$

解答作业 4. Consider the transform φ: $x = (\rho \cos^2 \phi)^{1/p}$, $y = (\rho \sin^2 \phi)^{1/q}$. The determinant of the Jacobian is $|\varphi'(\rho, \phi)| = \rho^{1/p+1/q-1} \cos^{2/p-1} \phi \sin^{2/q-1} \phi$. Define $\Omega := \{(\rho, \phi) \mid 0 \le \phi \le \pi/2, \quad 0 \le \rho^{1/p} \cos^{2/p} + \rho^{1/q} \sin^{2/q} \le 1\}$ Then we have

$$\begin{aligned}
&\iint_{0<|x|+|y|\le 1} \frac{dxdy}{|x|^p + |y|^q} = \iint_{(\rho,\theta)\in\Omega} \rho^{1/p+1/q-2} \cos^{2/p-1} \phi \sin^{2/q-1} \phi d\rho d\phi \\
= &\left(\int_{0 \le \rho^{1/p} \cos^{2/p} + \rho^{1/q} \sin^{2/q} \le 1} \rho^{1/p+1/q-2} d\rho\right) \left(\int_0^{\pi/2} \cos^{2/p-1} \phi \sin^{2/q-1} \phi d\phi\right).
\end{aligned}$$

Therefore, the singular integral $\iint\limits_{0<|x|+|y|\leq 1} \frac{dxdy}{|x|^p+|y|^q}$ converges under the condition $1/p+1/q \geq 1$ (first term) and $p>0.q>0$ (second term).

解答作业 5. a)

$$\begin{aligned}\int_0^\infty \cos x^2 dx &= \int_0^1 \frac{\cos t}{2\sqrt{t}}dt + \int_1^\infty \frac{\cos t}{2\sqrt{t}}dt \\ &= \sqrt{t}\cos t|_0^1 + \int_0^1 \sqrt{t}\sin t dt + \int_1^\infty \frac{\cos t}{2\sqrt{t}}dt.\end{aligned}$$

Since the second and third terms are convergent, the whole singular integral converges.

b) Not convergent. However, I guess there is a type here, it should be $\cos x^2$ instead of $\cos^2 x$. If so, the singular integral is convergent.

c)

$$\begin{aligned}\lim_{n\to\infty} \iint\limits_{|x|\leq n} \sin(x^2+y^2)dxdy &= 4\lim_{n\to\infty}\int_0^n dx \lim_0^\infty(\sin x^2\cos y^2+\cos x^2\sin y^2)dy \\ &= 8\left(\lim_{n\to\infty}\int_0^n \sin x^2 dx\right)\left(\lim_{n\to\infty}\int_0^n \cos x^2 dx\right) = 8\cdot\left(\frac{\sqrt{\pi}}{2\sqrt{2}}\right)^2 = \pi.\end{aligned}$$

Here the last equality uses the Fresnel integral (Homework 15/13(b), Last term).

$$\iint\limits_{x^2+y^2\leq 2n\pi} \sin(x^2+y^2)dxdy = 0 = \iint\limits_{\rho^2\leq 2n\pi} \rho\sin\rho^2 d\rho d\phi = \pi \int\limits_{0\leq\rho^2\leq 2n\pi} \sin r^2 dr^2 = 0.$$

Finally, the last comment can be verified in the same manner as Homework 11/7.

解答作业 6. First

$$\int_{-1}^1 \frac{h}{h^2+x^2}dx = \int_{-1/h}^{1/h}\frac{1}{1+t^2}dt = 2\arctan\frac{1}{h}.$$

Therefore, it holds that

$$\lim_{h\to 0}\frac{1}{\pi}\int_{-1}^1 \frac{h}{h^2+x^2}\cdot A dx = \frac{1}{\pi}A\pi = A,$$

for any given constant A. Let $|f(x)|\leq M$ for any $x\in[-1,1]$. On the other hand, for any $\epsilon>0$, there exists $\delta\leq 1$ such that

$$|f(x)-f(x)|\leq\epsilon.$$

Therefore,

$$\begin{aligned}\frac{1}{\pi}\int_{-1}^1 \frac{h}{h^2+x^2}f(x)dx &= \frac{1}{\pi}\int_{x\in[-1,1]\setminus[-\delta,\delta]}\frac{h}{h^2+x^2}f(x)dx + \frac{1}{\pi}\int_{-\delta}^{\delta}\frac{h}{h^2+x^2}f(x)dx \\ \leq & \frac{1}{\pi}\int_{x\in[-1,1]\setminus[-\delta,\delta]}\frac{h}{h^2+x^2}Mdx + \frac{1}{\pi}\int_{-\delta}^{\delta}\frac{h}{h^2+x^2}(f(0)+\epsilon)dx\end{aligned}$$

Taking limits on both sides, we obtain

$$\lim_{h\to 0}\frac{1}{\pi}\int_{-1}^{1}\frac{h}{h^2+x^2}f(x)dx \le 0+(f(0)+\epsilon)=f(0)+\epsilon.$$

We can obtain

$$\lim_{h\to 0}\frac{1}{\pi}\int_{-1}^{1}\frac{h}{h^2+x^2}f(x)dx \ge f(0)-\epsilon$$

in the same manner. According to arbitrary of ϵ, we complete the proof.

解答作业 7.

$$\begin{aligned}\int_1^\infty dx\int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}dy &= \int_1^\infty dx\int_{1/x}^\infty \frac{1-t^2}{x(1+t^2)^2}dt\\ = \quad \int_1^\infty \frac{1}{x}dx\int_{1/x}^\infty\left[\frac{2}{(1+t^2)^2}-\frac{1}{1+t^2}\right]dt &= -\frac{\pi}{4}. \end{aligned} \tag{4}$$

Similarly, we obtain

$$\int_1^\infty dy\int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}dx = \frac{\pi}{4}. \tag{5}$$

Denote $E_i^x=\{(x,y)\mid 1\le x\le i, y\ge 1\}$, $E_i^y=\{(x,y)\mid x\ge 1, 1\le y\le i\}$. Clearly, if

$$\iint\limits_{x\ge 1, y\ge 1} f(x,y)dxdy$$

exists, then it holds,

$$\lim_{n\to\infty}\iint_{E_n^x} f(x,y)dxdy = \iint\limits_{x\ge 1, y\ge 1} f(x,y)dxdy = \lim_{n\to\infty}\iint_{E_n^y} f(x,y)dxdy.$$

It follows the relationships (4) and (5) that the right hand side and left hand side of the above equalities are not equal. This completes the proof.